Math 309 Quiz 3

November 19, 2015

Problem 1. Write down the definition of the fundamental period of a periodic function $f(x)$.

Solution 1. The fundamental period is the smallest positive value of T for which $f(x+T) = f(x)$ for all x.

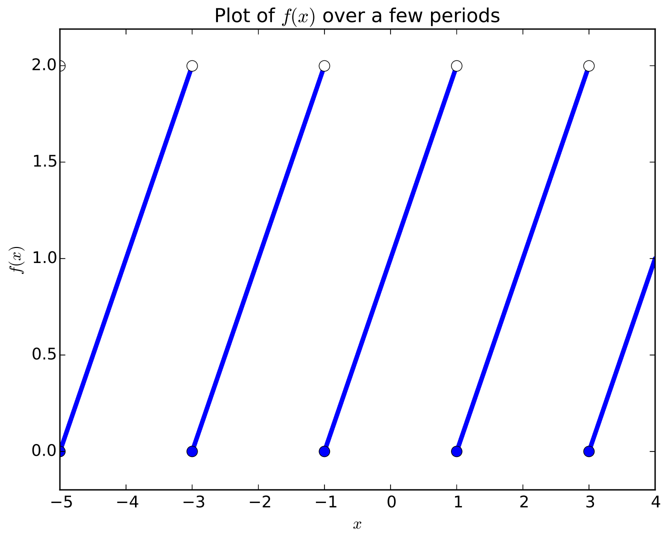
Problem 2. Consider the function defined by $f(x) = x$ for $-1 \leq x < 1$, and with $f(x+2) = f(x)$ for all x.

(a) Sketch a graph of the function that includes two full periods

(b) Find the Fourier series for $f(x)$.

Solution 2.

(a) A plot of the function is included below



(b) We next calculate the Fourier series of the function. Since $f(x)$ is odd, we know that the a_n terms will all be zero. (Again, this is because the integral of an odd function over a symmetric domain is zero). Therefore we need only calculate the b_n terms. Note also that the fundamental period of our function is 2. Using the Euler-Fourier formula we have that:

$$b_n(x) = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right).$$

Now since the fundamental period is 2, we take L to be 1 in the above. Furthermore since $f(x)$ is odd and the product of two odd functions is even, we can multiply by 2 and just integrate from 0 to L. Thus

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x)\sin(n\pi x)dx = 2\int_0^1 f(x)\sin(n\pi x)dx \\ &= 2\int_0^1 x\sin(n\pi x)dx = -\frac{2}{n\pi}x\cos(n\pi x)|_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)dx \\ &= -\frac{2}{n\pi}\cos(n\pi) + \frac{2}{n^2\pi^2}\sin(n\pi x)|_0^1 = -\frac{2}{n\pi}\cos(n\pi). \end{aligned}$$

Therefore since $\cos(n\pi) = (-1)^n$, we have that $b_n = \frac{2}{n\pi}(-1)^{n+1}$, and therefore

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x).$$