Math 309 Quiz 4

November 19, 2015

Problem 1. Consider the function

$$f(x) = \begin{cases} 0 & -2 \le x < 0 \\ 1 & 0 \le x < 2 \end{cases}, \quad f(x+4) = f(x) \text{ for all } x.$$

(a) Find the Fourier series of $f(x)$

(b) For which values of x does the Fourier series converge to $f(x)$?

(c) What does the Fourier series converge to at the remaining points?

Solution 1.

(a) First note that the fundamental period of $f(x)$ is 4. Therefore in the Euler-Fourier formula, we should take $L = 2$. We then calculate

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos(n\pi x/2)dx \\ &= \frac{1}{2}\int_{0}^{2}\cos(n\pi x/2)dx = \frac{1}{n\pi}\sin(n\pi x/2)|_0^2 = 0. \end{aligned}$$

This calculation works for all n except for $n = 0$, since we are dividing by n in the calculation. Therefore we need to calculate a_0 separately. We have that

$$a_0 = \int_1 L \in_{-L}^{L} f(x)dx = \frac{1}{2}\int_0^2 dx = 1.$$

Furthermore, we calculate

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin(n\pi x/2)dx$$
$$= \frac{1}{2}\int_{0}^{2}\sin(n\pi x/2)dx = -\frac{1}{n\pi}\cos(n\pi x/2)|_0^2 = -\frac{1}{n\pi}(\cos(n\pi)-1) = -\frac{1}{n\pi}((-1)^n-1).$$

We note that b_n is zero whenever n is even. Therefore only a_0 and the odd b_n terms are nonzero. This gives

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_{2n-1}\sin((2n-1)\pi x/2) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin((2n-1)\pi x/2).$$

(b) We know that the Fourier series of $f(x)$ converges to $f(x)$ exactly at the points where $f(x)$ is continuous, by the pointwise convergence theorem we discussed in class. The points where $f(x)$ is discontinuous are exactly the even integers. Therefore the Fourier series converges to $f(x)$ for all values of x except for the even integers.

(c) By our pointwise convergence theorem, we know that at the discontinuities of $f(x)$, the Fourier series converges to $(f(x+)+f(x-))/2$, eg. the average of the left and right limits of $f(x)$ at that point. At the points of discontinuity for our function $f(x)$, the limits are 0 and 1. Therefore at even integers, our Fourier series converges to the value $1/2$.

Problem 2. Consider the function

$$f(x) = \frac{1}{2} + \sin(3x) + \cos(4x) + 3\cos(7x).$$

Calculate the value of

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx.$$

[Hint: you can use Parseval's identity (though it's not necessary]

Solution 2. If we recall Parseval's identity:

$$\frac{1}{L}\int_{-L}^{L} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$

and recognize that for our function $f(x)$, we have that $L = \pi$, and that $a_0 = 1, b_3 = 1, a_4 = 1, b_7 = 3$, and a_i, b_j are zero otherwise, then this says

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = \frac{1}{2} + 1^2 + 1^2 + 3^2 = 11.5.$$

ALTERNATIVE SOLUTION: We can also calculate $\int f(x)^2 dx$ using the orthogonality of sines and cosines. Note that

$$f(x)^2 = \left(\frac{1}{2}\right)^2 + \sin(3x)^2 + \cos(4x)^2 + (3\cos(7x))^2 + \text{cross terms}.$$

By orthogonality, the integral from $-\pi$ to π of all of the cross terms will be zero! Therefore

$$\int_{-\pi}^{\pi} f(x)^2 dx = \int_{-\pi}^{\pi} \left(\frac{1}{2}\right)^2 dx + \int_{-\pi}^{\pi} \sin(3x)^2 dx + \int_{-\pi}^{\pi} \cos(4x)^2 dx + \int_{-\pi}^{\pi} (3\cos(7x))^2 dx$$

Now we note that

$$\int_{-L}^{L} \cos^2(n\pi x/L) dx = \int_{-L}^{L} ((1/2)+(1/2)\cos(2n\pi x/L) dx = \int_{-L}^{L} ((1/2)) dx = L,$$

and also

$$\int_{-L}^{L} \sin^2(n\pi x/L) dx = \int_{-L}^{L} ((1/2)-(1/2)\cos(2n\pi x/L) dx = \int_{-L}^{L} ((1/2)) dx = L,$$

Therefore

$$\int_{-\pi}^{\pi} f(x)^2 dx = 2\pi \left(\frac{1}{2}\right)^2 + \pi + \pi + 9\pi = 11.5\pi.$$

Thus

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = 11.5$$