Math 309 Quiz 5

November 24, 2015

Problem 1. Assuming that $\lambda > 0$, determine for which values of λ the BVP

$$\begin{cases} \psi''(x) + \lambda\psi(x) = 0 \\ \psi(0) = 0, \ \psi(L) = 0 \end{cases}$$

has a nontrivial solution. SHOW YOUR WORK!!

Solution 1. The general solution to $\psi''+\lambda\psi = 0$ is obtained from an analysis of the roots of the characteristic polynomial $r^2 + \lambda$. These roots are $\pm\sqrt{-\lambda}$, and since $\lambda > 0$ these are actually $\pm i\sqrt{\lambda}$. This points to a general solution of the form

$$\psi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Now since $\psi(0) = 0$, we have that $A = 0$. Furthermore, since $\psi(L) = 0$ we have $B\sin(\sqrt{\lambda}L) = 0$. If $B = 0$, then this gives us a trivial solution – since we're looking for λ such that there exists a nontrivial solution, we may therefore assume that $B \neq 0$. Then $\sin(\sqrt{\lambda}L) = 0$, and this implies $\sqrt{\lambda}L = n\pi$ for some integer n. Therefore $\lambda = n^2\pi^2/L^2$ for n a nonzero integer integer n.

Problem 2. Find a solution to the heat equation

$$\begin{cases} u_t - 7u_{xx} = 0, & 0 \leq x \leq 3, t > 0 \\ u(0,t) = 0, \quad u(3,t) = 0, & t > 0u(x,0) = \sin(\pi x) \end{cases}$$

Solution 2. We wish to use our ideas about the fundamental solutions to the heat equation, but in order to do so we must first identify the values of α^2 and L. Since the domain of interest is $0 \leq x \leq 3$, we see that $L = 3$. Moreover it's clear from the heat equation specified that $\alpha^2 = 7$. The fundamental solutions we have are therefore

$$u_n(x,t) = e^{-(n^2\pi^2\alpha^2/L^2)t}\sin(n\pi x/L) = e^{-n^2\pi^2(7/9)t}\sin(n\pi x/L)$$

Now to finish up, we need to figure out the Fourier coefficients of our initial function $u(x,0) = \sin(\pi x)$. We are trying to write this as a linear combination of functions of the form $\sin(n\pi x/3)$. This is easy to do, since $\sin(\pi x) = \sin(n\pi x/3)$ for $n = 3$. Therefore the solution we are looking for is

$$u(x,t) = u_3(x,t) = e^{-7\pi^2 t}\sin(\pi x).$$