Math 309 Quiz 6

December 9, 2015

Problem 1. Find the asymptotic (steady state) solution to the heat equation problem:

$$u_t - 42u_{xx} = 0$$
$$u(0,t) = 0, u(2,t) = 14$$
$$u(x,0) = 0,\ 0 < x < 2$$

Solution 1. This problem might feel a bit confusing if one actually tries to desteadyine the value of $u(x,t)$ for all times t. This is because the sine expansion of 0 is 0, so our usual method of separation of variables doesn't seem to work so well. However, physically we're starting with a completely cold bar and we're heating one end, so eventually *some* heat will leak into the bar. Since we're only interested in the asymptotic behavior, we really don't care what this looks like during intermediate times. What will it look like at late times though? Eventually $u(x,t)$ will approach a time independent value $u_{\text{steady}}(x)$. It will still be a solution to the heat equation, and this means that

$$u''_{\text{steady}}(x) = 0.$$

Therefore $u_{\text{steady}} = ax + b$ for some constants a and b. Now since the steady state distribution should satisfy the boundary conditions, we know that $u_{\text{steady}}(0) = 0$ and $u_{\text{steady}}(2) = 14$, and therefore $a = 7$ and $b = 0$. Thus $u_{\text{steady}}(x) = 7x$.

Problem 2. Find the solution to the heat equation

$$u_t - 4u_{xx} = 0$$
$$u_x(0,t) = 0, \quad u_x(2\pi, t) = 0$$
$$u(x,0) = \cos(x) - 4\cos(3x/2)$$

Solution 2. We have von Neumann boundary conditions, and this tells us to take the cosine expansion of the initial function. The value of L is 2π, so we are trying to write

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx/2).$$

This is accomplised by taking $a_2 = 1, a_3 = -4$, and $a_n = 0$ for all other values of n. Using this, we see that the solution is

$$u(x,t) = u_2(x,t) - 4u_3(x,t),$$

where the fundamental solutions are given by

$$u_n(x,t) = e^{-n^2\pi^2\alpha^2 t/L^2} \cos(n\pi x/L) = e^{-n^2 t} \cos(nx/2).$$

From this, we see that

$$u(x,t) = e^{-4t} \cos(x) - 4e^{-9t} \cos(3x/2).$$