Math 309 Quiz 7

December 9, 2015

Problem 1. Use separation of variables to reduce the modified wave equation

$$u_{tt} + \gamma u - c^2 u_{xx} = 0$$

into a system of two ordinary differential equations.

Solution 1. We assume that $u(x,t) = F(x)T(t)$ for some functions $F(x)$ and $T(t)$. Then we have that

$$F(x)T''(t) + \gamma F(x)T(t) - c^2 F''(x)T(t) = 0.$$

Dividing everything by $F(x)T(t)$, we obtain:

$$T''(t)/T(t) + \gamma - c^2 F''(x)/F(x) = 0.$$

Therefore

$$T''(t)/T(t) + \gamma = c^2 F''(x)/F(x).$$

Now the left hand side is a function of t only and the right hand side is a function of x only, and therefore both are equal to a constant $-\lambda$. Thus

$$T''(t)/T(t) + \gamma = -\lambda$$

and also

$$c^2 F''(x)/F(x) = -\lambda$$

Simplifying things, we obtain:

$$T''(t) + (\gamma + \lambda)T(t) = 0$$

as well as

$$F''(x) + (\lambda/c^2)F(x) = 0.$$

Problem 2. Prove that

$$u(x,t) = f(x - ct), \text{ and } u(x,t) = f(x + ct)$$

are both solutions to the wave equation $u_{tt} - c^2 u_{xx} = 0$.

Solution 2. Suppose that $u(x,t) = f(x \pm ct)$. Using the chain rule, we calculate $u_x(x,t) = f'(x \pm ct)$ and $u_{xx}(x,t) = f''(x \pm ct)$, as well as $u_t(x,t) = f'(x \pm ct)(\pm c)$ and $u_{tt} = f''(x \pm ct)(\pm c)^2$. Using this we calculate

$$u_{tt} - c^2 u_{xx} = f''(x \pm ct)(\pm c)^2 - c^2 f''(x \pm ct) = 0.$$

Thus $u(x,t)$ is a solution to the heat equation.