Math 309 Quiz 9

December 17, 2015

Before solving any of these problems, it's helpful to consider in general solutions to Laplace's equation in the rectangle bounded by $x = 0, y = 0, x = L, y = M$. If three out of four of the walls have a homogeneous boundary condition, then there exists an infinite set of solutions, which we call the fundamental set of solutions satisfying this homogeneous boundary data. If on the fourth wall we have some nonhomogeneous boundary data, then the solution we are looking for satisfying that nonhomogeneous boundary data is obtained by taking an appropriate linear combination of the fundamental set of solutions. However, we should make one very important observation: the fundamental set of solutions we have depends on which of the four walls has the nonhomogeneous condition!!!! One can work out, using separation of variables, the fundamental set of solutions in each of the four possible cases. The solution you get in each is as follows.

North Wall: If the nonhomogeneous boundary is the North wall ($y = M$), then the fundamental set of solutions is

$$u_n(x, y) = \sin(n\pi x/L)\sinh(n\pi y/L),\ n = 1, 2, 3, \ldots$$

South Wall: If the nonhomogeneous boundary is the South wall ($y = 0$), then the fundamental set of solutions is

$$u_n(x, y) = \sin(n\pi x/L)\sinh(n\pi(y - M)/L),\ n = 1, 2, 3, \ldots$$

West Wall: If the nonhomogeneous boundary is the West wall ($x = 0$), then the fundamental set of solutions is

$$u_n(x, y) = \sinh(n\pi(x - L)/M)\sin(n\pi y/M),\ n = 1, 2, 3, \ldots$$

East Wall: If the nonhomogeneous boundary is the East wall ($x = L$), then the fundamental set of solutions is

$$u_n(x,y) = \sinh(n\pi x/M)\sin(n\pi y/M),\ n = 1, 2, 3, \ldots$$

Problem 1. Find a solution to Laplace's equation $u_{xx} + u_{yy} = 0$ in the rectangle bounded by the lines $x = 0, x = 2, y = 0$, and $y = 1$, and satisfying the boundary conditions

$$u(0,y) = 0,\ u(2,y) = 1 - |2y-1|,\ 0 \le y \le 1$$

$$u(x,0) = 0,\ u(x,1) = 0,\ 0 \le x \le 2.$$

Solution 1. Since $L = 2$ and $M = 1$, and the nonhomogeneous boundary is the East wall, we look for a solution of the form

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi x)\sin(n\pi y).$$

for some constants c_n. Then from the nonhomogeneous boundary, we see that

$$1 - |2y-1| = u(2,y) = \sum_{n=1}^{\infty} c_n \sinh(2n\pi)\sin(n\pi y).$$

Now calculating the sine series of $1 - |2y - 1|$, we see that

$$1 - |2y-1| = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\sin(n\pi/2)\sin(n\pi y), \quad 0 \le y \le 1.$$

Therefore we should take

$$c_n = \frac{4}{n^2\pi^2}\frac{\sin(n\pi/2)}{\sinh(2n\pi)},$$

giving us

$$u(x,y) = \sum_{n=1}^{\infty} c_n u_n(x,y) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\frac{\sin(n\pi/2)}{\sinh(2n\pi)}\sinh(n\pi x)\sin(n\pi y).$$

Problem 2. Find a solution to Laplace's equation $u_{xx} + u_{yy} = 0$ in the rectangle bounded by the lines $x = 0, x = 2, y = 0$, and $y = 3$, and satisfying the boundary conditions

$$u(0, y) = 0,\ u(2, y) = 0,\ 0 \le y \le 1$$

$$u(x, 0) = \sin(\pi x),\ u(x, 3) = \sin(\pi x/2),\ 0 \le x \le 2.$$

Solution 2. From the above, we know what to do in the case that three out of four boundaries are homogeneous. However, in our problem, we are faced with a conundrum: there is more than one nonhomogeneous boundary condition! How can we handle this? We break up the problem into two parts, and use the superposition principle!

Subproblem 1: We find a solution to Laplaces equation satisfying

$$u(0, y) = 0,\ u(2, y) = 0,\ 0 \le y \le 1$$

$$u(x, 0) = 0,\ u(x, 3) = \sin(\pi x/2),\ 0 \le x \le 2.$$

To do so, note that $L = 2, M = 3$ and the nonhomogeneous wall for this is the North wall, so we should look for a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/2) \sinh(n\pi y/2).$$

The last boundary condition says that $u(x, 3) = \sin(\pi x/2)$, which is satisfied if we take $c_1 = 1/\sinh(3\pi/2)$ and $c_n = 0$ otherwise. Therefore the solution is

$$u(x, y) = \frac{1}{\sinh(3\pi/2)} \sin(\pi x/2) \sinh(\pi y/2).$$

Subproblem 2: We find a solution to Laplaces equation satisfying

$$u(0, y) = 0,\ u(2, y) = 0,\ 0 \le y \le 1$$

$$u(x, 0) = \sin(\pi x),\ u(x, 3) = 0\ 0 \le x \le 2.$$

To do so, note that $L = 2, M = 3$ and the nonhomogeneous wall for this is the South wall, so we should look for a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/2) \sinh(n\pi (y - 1)/2).$$

The last boundary condition says that $u(x,0) = \sin(\pi x)$, which is satisfied if we take $c_2 = 1/\sinh(-\pi)$ and $c_n = 0$ otherwise. Therefore the solution is

$$u(x,y) = \frac{1}{\sinh(-\pi)} \sin(\pi x) \sinh(\pi(y-1)).$$

Putting it together: If we add the two solutions we found together, the superposition principle tells us that it is also a solution to Laplaces equation. Moreover, the sum satisfies the desired boundary conditions. Therefore the solution to the original problem is

$$u(x,y) = \frac{1}{\sinh(3\pi/2)} \sin(\pi x/2) \sinh(\pi y/2) + \frac{1}{\sinh(-\pi)} \sin(\pi x) \sinh(\pi(y-1)).$$