A Primer on Complex Numbers

Intro. to Differential Equations

October 21, 2015

1 Complex Numbers

Definition 1. A *complex number* is a number that may be expressed in the form

$$a + bi$$

for some real numbers a and b. The value a is called the *real part* of the complex number $a + ib$, and the value b is called the *imaginary part*.

Here i is a special number satisfying $i^2 = -1$. In other words, i is a square root of -1. Note that the real and imaginary parts of a complex number are both real. We use $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ to denote the real and imaginary parts of a complex number z. In particular:

$$\mathrm{Re}(a + ib) = a, \quad \mathrm{Im}(a + ib) = b.$$

Exercise 1. Determine the real and imaginary components of the followng numbers

(a) 13

(b) $-7i$

(c) $3 - 2i$

(d) $-4 + 7i$

The set of complex numbers forms a field. We can add, subtract, multiply, and divide (assuming nonzero denominator) complex numbers, and each of these operations behaves very much in the same way as over the real numbers (commutative, associative, distributive, etc).

Addition/Subtraction:
To add two complex numbers $a + ib$ and $c + id$, one simply adds the real and imaginary components:

$$(a + ib) + (c + id) = (a + c) + i(b + d).$$

Subtraction is similar:

$$(a + ib) - (c + id) = (a - c) + i(b - d).$$

Multiplication:
To multiply two complex numbers $a + ib$ and $c + id$, one simply "foils it out":

$$(a + ib)(c + id) = ac + ibc + iad + i^2bd = ac - bd + i(ad + bc).$$

Division of Complex Numbers:
The operation of division for complex numbers is much more interesting. Given complex numbers $a + ib$ and $c + id$, with the latter nonzero, we can calculate $(a + ib)/(c + id)$ via the following complex conjugation trick:

$$\frac{a + ib}{c + id} = \frac{a + ib}{c + id}1 = \frac{a + ib}{c + id}\frac{c - id}{c - id} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2} = \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{bc - ad}{c^2 + d^2}\right).$$

The above trick amounts to multiplying and dividing by the complex conjugate of the denominator.

Definition 2. Let $z = a + ib$, be a complex number. The *complex conjugate* of z is denoted $\overline{z}$ and given by

$$\overline{z} = a - ib.$$

In other words the complex conjugate of a complex number is the unique complex number with the equal real part and negative imaginary part. Note that

$$z\overline{z} = (a + ib)(a - ib) = a^2 - iab + iab - i^2b^2 = a^2 - i^2b^2 = a^2 + b^2.$$

In particular a complex number times its complex conjugate is purely real!

Definition 3. Let $z = a + ib$ be a complex number. The the *norm* (or modulus) of z is

$$|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}.$$

Exercise 2. Write each of the following expressions as a complex number in the form $a + ib$ for a and b real.

(a) $(3 + 7i) + (4 - 2i)$

(b) $(2 + 3i) - (1 + 3i)$

(c) $(3 + 6i)(1 - 4i)$

(d) $(1 + 2i)/(1 + 7i)$

Thinking geometrically, we can view a complex number $a + ib$ as the vector $\langle a, b\rangle$ in the x, y-plane. Recall that for vectors in the plane, we have an alternative description in terms of the magnitude of the vector, and the angle from the positive x-axis. Note that the magnitude of the vector $\langle a, b\rangle$ is exactly the magnitude of the complex number $a + ib$.

Definition 4. Consider a complex number $z = a + ib$, and let θ be the angle from the positive x-axis to the vector $\langle a, b\rangle$ in the x, y-plane. Then θ is called the *argument* of z, and is denoted by $\text{Arg}(z)$.

Since the vector $\langle a, b\rangle$ is precisely determined by it's magnitude and angle, each complex number is uniquely determined by its magnitude and argument.

Exercise 3. Find the magnitude and argument of each of the following complex numbers

(a) $1+i$

(b) $1-i$

(c) $3+4i$

2 Euler's Definition

Integral to our understanding of the complex numbers is the following definition of Euler

Definition 5 (Euler)**.** For any real number θ, we define the complex number $e^{i\theta}$ by

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Then the big theorem of Euler is the following

Theorem 1. *This definition does not break anything.*

In other words, the definition agrees with the usual algebraic identities of exponential funtions, ie.

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}.$$

As well as the usual power series representation for the exponential function

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}.$$

As a matter of fact, any complex number $z = a+ib$, if $r = \sqrt{a^2+b^2} = |z|$ and $\theta = \mathrm{Arg}(z)$, then $z = re^{i\theta}$. Thus for any complex number, we can go back and forth between the cartesian $a+ib$ form and the exponential $re^{i\theta}$ form.

Exercise 4. Write each of the following complex numbers in the form $re^{i\theta}$ for some real numbers r and θ with $r \geq 0$.

(a) i

(b) $2+2i$

(c) $3+4i$

Exercise 5. Write each of the following complex numbers in the form $a+ib$ for some real values a and b.

1. $e^{i\pi}$
2. $2e^{i\pi/6}$
3. $-1e^{i5\pi/4}$
4. $e^{\ln(2)+i\pi}$

3 Applications

Proposition 1. *Let θ and ϕ be two real numbers. Then*

$$\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi),$$

$$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi).$$

Proof. From Euler's definition,

$$\sin(\theta+\phi) = \mathrm{Im}(e^{i(\theta+\phi)}),$$

and also

$$\cos(\theta+\phi) = \mathrm{Re}(e^{i(\theta+\phi)}).$$

Now again from Euler, we calculate:

$$\begin{aligned} e^{i(\theta+\phi)} &= e^{i\theta+i\phi} = e^{i\theta}e^{i\phi} \\ &= (\cos(\theta) + i\sin(\theta))(\cos(\phi) + i\sin(\phi)) \\ &= \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) + i(\sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)). \end{aligned}$$

From this, it follows that

$$\mathrm{Im}(e^{i(\theta+\phi)}) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$

and also that

$$\mathrm{Re}(e^{i(\theta+\phi)}) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi).$$

This completes the proof. □

Proposition 2. *Let a and b be real numbers. Then*

$$\begin{aligned} \int e^{at}\sin(bt)dt &= \frac{-b}{a^2+b^2}e^{at}\cos(bt) + \frac{a}{a^2+b^2}e^{at}\sin(bt) + C \\ \int e^{at}\cos(bt)dt &= \frac{a}{a^2+b^2}e^{at}\cos(bt) + \frac{b}{a^2+b^2}e^{at}\sin(bt) + C \end{aligned}$$

Proof. Note that by Euler,

$$\begin{aligned} e^{(a+ib)t} &= e^{at+ibt} = e^{at}e^{ibt} \\ &= e^{at}(\cos(bt) + i\sin(bt)) \\ &= e^{at}\cos(bt) + ie^{at}\sin(bt). \end{aligned}$$

It follows that

$$\mathrm{Im}(e^{(a+ib)t}) = e^{at}\sin(bt),$$

and also that

$$\mathrm{Re}(e^{(a+ib)t}) = e^{at}\cos(bt).$$

Now taking real or imaginary parts commutes with integration, and therefore

$$\begin{aligned}\int e^{at}\sin(bt)dt &= \int \mathrm{Im}(e^{(a+ib)t})dt \\ &= \mathrm{Im}(\int e^{(a+ib)t}dt) \\ &= \mathrm{Im}(\frac{1}{(a+ib)}e^{(a+ib)t})\end{aligned}$$

Where for simplicity we have left off the usual arbitrary constant of integration. Now using our complex conjugation trick and Euler's definition:

$$\begin{aligned}&\frac{1}{(a+ib)}e^{(a+ib)t} \\ &= \frac{1}{(a+ib)}\frac{a-ib}{a-ib}e^{(a+ib)t} \\ &= \left(\frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}\right)\left(e^{at}\cos(bt) + ie^{at}\sin(bt)\right) \\ &= \left(\frac{a}{a^2+b^2}e^{at}\cos(bt) + \frac{b}{a^2+b^2}e^{at}\sin(bt)\right) \\ &+ i\left(\frac{-b}{a^2+b^2}e^{at}\cos(bt) + \frac{a}{a^2+b^2}e^{at}\sin(bt)\right)\end{aligned}$$

Thus

$$\int e^{at}\sin(bt)dt = \mathrm{Im}\left(\frac{1}{(a+ib)}e^{(a+ib)t}\right) = \left(\frac{-b}{a^2+b^2}e^{at}\cos(bt) + \frac{a}{a^2+b^2}e^{at}\sin(bt)\right).$$

and similarly

$$\int e^{at}\cos(bt)dt = \mathrm{Re}\left(\frac{1}{(a+ib)}e^{(a+ib)t}\right) = \left(\frac{a}{a^2+b^2}e^{at}\cos(bt) + \frac{b}{a^2+b^2}e^{at}\sin(bt)\right).$$

□

Exercise 6. Use Euler to prove the trigonometric identity

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos(\alpha)\cos(\beta).$$

Exercise 7. The hyperbolic trigonometric functions $\sinh(x)$ and $\cos(x)$ are defined by

$$\sinh(x) = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x},$$

and

$$\cosh(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}.$$

(a) Use Euler to show that

$$\sinh(x) = -i\sin(ix), \quad \cosh(x) = \cos(ix)$$

This explains their designation as "trigonometric functions".

(b) Using (a), prove that

$$\frac{d}{dx}\sinh(x) = \cosh(x), \quad \frac{d}{dx}\cosh(x) = \sinh(x)$$

One can obviously also prove this directly from the definitions, but it's even easier this way!

Exercise 8. Use Euler's definition to calculate the integral

$$\int xe^{3x}\cos(2x)dx.$$