

Matrix Exponential Formulas

Linear Analysis

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Abstract

We present some tricks for quickly calculating the matrix exponential of certain special classes of matrices, such as 2×2 matrices.

1 2×2 matrix exponential formulas

In this section, we address how to quickly calculate a matrix exponential by hand for a 2×2 matrix A . The answer, it turns out, will depend on the particulars of the eigenvalues of A . There are three possible cases

- (a) the eigenvalues of A are real and distinct
- (b) the eigenvalues of A are real and the same
- (c) the eigenvalues of A are a complex conjugate pair

We already know how to calculate $\exp(At)$ in any of these cases – we conjugate A to get a matrix in Jordan normal form, we take the exponential of that matrix, and we conjugate back. This process can feel a bit tedious, since it involves finding the eigenspaces for each eigenvalue, possibly finding generalized eigenvectors, calculating products of matrices, etc. Moreover, it can sometimes feel that to calculate several matrix exponentials by hand without creating some sort of algebraic catastrophe along the way will require a small miracle. Thus we are motivated to determine simple explicit equations for the value of A in each of the above cases. We do so below, and our results are summarized in the following table Using the above table, if we figure

Eigenvalues of A	$\exp(At)$
r_1, r_2 real and distinct	$e^{r_1 t} \frac{1}{r_1 - r_2} (A - r_2 I) - e^{r_2 t} \frac{1}{r_1 - r_2} (A - r_1 I)$
r repeated twice	$e^{rt} I + e^{rt} t (A - rI)$
$a \pm ib$ complex conjugate pair	$e^{at} \cos(bt) I + \frac{1}{b} e^{at} \sin(bt) (A - aI)$

out the eigenvalues of A , then we can insert them into the relevant formula above to obtain the desired matrix exponential, without the bother of diagonalization, Jordan normal form, etc.

Example 1. Consider the matrix

$$A = \begin{pmatrix} -27 & 10 \\ -75 & 28 \end{pmatrix}$$

We calculate the characteristic polynomial of A to be $p_A(x) = x^2 - x - 6 = (x-3)(x+2)$. Therefore the eigenvalues are real and distinct, given by $r_1 = 3, r_2 = -2$. We then calculate

$$\begin{aligned} \exp(At) &= e^{r_1 t} \frac{1}{r_1 - r_2} (A - r_2 I) - e^{r_2 t} \frac{1}{r_1 - r_2} (A - r_1 I) \\ &= e^{3t} \frac{1}{5} (A + 2I) - e^{-2t} \frac{1}{5} (A - 3I) \\ &= e^{3t} \frac{1}{5} \begin{pmatrix} 29 & 10 \\ -75 & 30 \end{pmatrix} - e^{-2t} \frac{1}{5} \begin{pmatrix} 24 & 10 \\ -75 & 25 \end{pmatrix} = \begin{pmatrix} (29/5)e^{3t} - (24/5)e^{-2t} & 2e^{3t} - 2e^{-2t} \\ -15e^{3t} + 15e^{-2t} & 6e^{3t} - 5e^{-2t} \end{pmatrix} \end{aligned}$$

Example 2. Consider the matrix

$$A = \begin{pmatrix} 6 & -1 \\ 9 & 0 \end{pmatrix}$$

We calculate the characteristic polynomial of A to be $p_A(x) = x^2 - 6x + 9 = (x-3)(x-3)$. Therefore the eigenvalue is $r = 3$, repeated twice. We then calculate

$$\begin{aligned} \exp(At) &= e^{rt} I + e^{rt} (A - rI) \\ &= e^{3t} I + e^{3t} (A - 3I) \\ &= e^{3t} I + e^{3t} t \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} = \begin{pmatrix} e^{3t} + 3te^{3t} & -te^{3t} \\ 9te^{3t} & e^{3t} - 3te^{3t} \end{pmatrix} \end{aligned}$$

Example 3. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 \\ -7 & 1 \end{pmatrix}$$

We calculate the characteristic polynomial of A to be $p_A(x) = x^2 - 2x + 29$. From the quadratic formula, we find that the roots are $1 \pm 2\sqrt{7}i$. This is the complex conjugate pair case $a \pm ib$ with $a = 1$ and $b = 2\sqrt{7}$. We then calculate

$$\begin{aligned} \exp(At) &= e^{at} \cos(bt) I + \frac{1}{b} e^{at} \sin(bt) (A - rI) \\ &= e^t \cos(2\sqrt{7}t) I + \frac{1}{2\sqrt{7}} e^t \sin(2\sqrt{7}t) (A - I) \\ &= e^t \cos(2\sqrt{7}t) I + \frac{1}{2\sqrt{7}} e^t \sin(2\sqrt{7}t) \begin{pmatrix} 0 & 4 \\ -7 & 0 \end{pmatrix} = \begin{pmatrix} e^t \cos(2\sqrt{7}t) & \frac{2}{\sqrt{7}} e^t \sin(2\sqrt{7}t) \\ \frac{-7}{2\sqrt{7}} e^t \sin(2\sqrt{7}t) & e^t \cos(2\sqrt{7}t) \end{pmatrix} \end{aligned}$$

1.1 Matrix Exponential for Distinct Real Values

Here we prove the equation in the table in the case that A has two real distinct eigenvalues r_1, r_2 . To this end, we set $a = (r_1 + r_2)/2$ and $b = (r_1 - r_2)/2$, so that

$r_1 = a + b$ and $r_2 = a - b$. Let P be a matrix which diagonalizes A so that $P^{-1}AP = D$ for $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Consider the matrix $B = A - aI$. We calculate

$$P^{-1}BP = P^{-1}(A - aI)P = P^{-1}AP - aI = \begin{pmatrix} \lambda_1 - a & 0 \\ 0 & \lambda_2 - a \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$$

Therefore we calculate

$$B^2 = P \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}^2 P^{-1} = P \begin{pmatrix} b^2 & 0 \\ 0 & b^2 \end{pmatrix} P^{-1} = P(b^2I)P^{-1} = b^2I.$$

Then since B and aI commute and $A = aI + B$, we see that

$$\exp(At) = \exp(aIt + Bt) = \exp(aIt) \exp(Bt) = e^{at} \exp(Bt).$$

So all we need to do now is calculate $\exp(Bt)$. Here we exploit our previous calculation that $B^2 = b^2I$. Using this, we see that $\frac{d^2}{dt^2} \exp(Bt) = B^2 \exp(Bt) = b^2 \exp(Bt)$. Therefore $\Psi(t) = \exp(Bt)$ is a solution to the (matrix) differential equation

$$\Psi''(t) = b^2 \Psi(t).$$

Any (matrix) solution to this equation is of the form

$$\Psi(t) = e^{bt}C_1 + e^{-bt}C_2$$

for some constant matrices C_1, C_2 . Therefore we know that

$$\exp(Bt) = e^{bt}C_1 + e^{-bt}C_2$$

for some constant matrices C_1, C_2 . We just need to figure out which ones! At $t = 0$, this says

$$I = C_1 + C_2.$$

Moreover, taking a derivative we get

$$B \exp(Bt) = be^{bt}C_1 - be^{-bt}C_2,$$

and at $t = 0$, this says

$$B = bC_1 - bC_2.$$

Putting this together, we see that

$$C_1 = \frac{1}{2}I + \frac{1}{2b}B, \quad C_2 = \frac{1}{2}I - \frac{1}{2b}B$$

Therefore

$$\exp(Bt) = e^{bt} \left(\frac{1}{2}I + \frac{1}{2b}B \right) + e^{-bt} \left(\frac{1}{2}I - \frac{1}{2b}B \right)$$

and it follows that

$$\begin{aligned} \exp(At) &= e^{at} \exp(Bt) = e^{(a+b)t} \left(\frac{1}{2}I + \frac{1}{2b}B \right) + e^{(a-b)t} \left(\frac{1}{2}I - \frac{1}{2b}B \right) \\ &= e^{r_1t} \left(\frac{1}{2}I + \frac{1}{r_1 - r_2}B \right) + e^{r_2t} \left(\frac{1}{2}I - \frac{1}{r_1 - r_2}B \right) \\ &= e^{r_1t} \left(\frac{1}{2}I + \frac{1}{r_1 - r_2}(A - aI) \right) + e^{r_2t} \left(\frac{1}{2}I - \frac{1}{r_1 - r_2}(A - aI) \right) \\ &= e^{r_1t} \frac{1}{r_1 - r_2}(A - r_2I) - e^{r_2t} \frac{1}{r_1 - r_2}(A - r_1I) \end{aligned}$$

This proves the accuracy of the first equation in the table.

1.2 Matrix Exponential for a Repeated Root

Here we prove the equation in the table in the case that A has one eigenvalue r repeated twice. In this case, the characteristic polynomial of A is $p_A(x) = x^2 - 2rx + r^2 = (x-r)^2$. By the Cayley-Hamilton theorem, we know that a matrix satisfies its characteristic polynomial. Therefore $(A - rI)^2 = 0$, and consequently $(A - rI)^k = 0$ for all $k \geq 2$. Therefore

$$\begin{aligned} \exp((A - rI)t) &= I + (A - rI)t + \frac{1}{2}(A - rI)^2t^2 + \frac{1}{6}(A - rI)^3t^3 + \frac{1}{4!}(A - rI)^4 + \dots \\ &= I + (A - rI)t + 0 + 0 + 0 + \dots \\ &= I + (A - rI)t. \end{aligned}$$

Furthermore, since $A - rI$ and rI commute, we have

$$\exp(At) = \exp((A - rI)t + rIt) = \exp((A - rI)t) \exp(rIt) = \exp((A - rI)t) e^{rt} I = e^{rt} I + e^{rt} t(A - rI).$$

This verifies the second equation in the table.

1.3 Matrix Exponential for Complex Conjugate Roots

Here we prove the formula in the table in the case that A has two eigenvalues which are complex conjugates $a \pm ib$. Let P be a matrix which diagonalizes A so that $P^{-1}AP = D$ for $D = \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix}$. Let B be the matrix $B = A - aI$. Then we calculate

$$P^{-1}BP = P^{-1}(A - aI)P = P^{-1}AP - aI = D - aI = \begin{pmatrix} ib & 0 \\ 0 & -ib \end{pmatrix}.$$

Therefore we see that

$$B^2 = P \begin{pmatrix} ib & 0 \\ 0 & -ib \end{pmatrix}^2 P^{-1} = P \begin{pmatrix} -b^2 & 0 \\ 0 & -b^2 \end{pmatrix} P^{-1} = P(-b^2I)P^{-1} = -b^2I.$$

Then since B and aI commute and $A = aI + B$, we see that

$$\exp(At) = \exp(aIt + Bt) = \exp(aIt) \exp(Bt) = e^{at} \exp(Bt).$$

So all we need to do now is calculate $\exp(Bt)$. Here we exploit our previous calculation that $B^2 = -b^2I$. Using this, we see that $\frac{d^2}{dt^2} \exp(Bt) = B^2 \exp(Bt) = -b^2 \exp(Bt)$. Therefore $\Psi(t) = \exp(Bt)$ is a solution to the (matrix) differential equation

$$\Psi''(t) = -b^2 \Psi(t).$$

Any (matrix) solution to this equation is of the form

$$\Psi(t) = \cos(bt)C_1 + \sin(bt)C_2$$

for some constant matrices C_1, C_2 . Therefore we know that

$$\exp(Bt) = \cos(bt)C_1 + \sin(bt)C_2$$

for some constant matrices C_1, C_2 . We just need to figure out which ones! At $t = 0$, this says

$$I = C_1,$$

Moreover, taking a derivative, we get

$$B \exp(Bt) = -b \sin(bt)C_1 + b \cos(bt)C_2,$$

and letting $t = 0$ this says

$$B = bC_2.$$

Therefore $C_1 = I$ and $C_2 = (1/b)B$ and

$$\exp(Bt) = \cos(bt)I + \sin(bt)\frac{1}{b}B.$$

Hence

$$\exp(At) = e^{at} \exp(Bt) = e^{at} \cos(bt)I + e^{at} \sin(bt)\frac{1}{b}B.$$

This verifies the third equation in the table.

2 $n \times n$ matrix exponential formulas

What about matrix exponential formulas for matrices larger than 2×2 ? There are tricks for these, too. One trick, if a matrix A is diagonalizable, is known as **Sylvester's Formula**. It's application is presented in the following theorem.

Theorem 1. *Suppose that A is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then*

$$\exp(At) = \sum_{i=1}^n e^{\lambda_i t} \prod_{j \neq i} \frac{1}{\lambda_i - \lambda_j} (A - \lambda_j I).$$

Thus for diagonalizable matrices, we may again immediately write down the matrix exponential in terms of the original matrix without fooling around with diagonalization. However, for this formula to work, it's very important that the matrix A be diagonalizable. For nondiagonalizable A , the situation is a bit more complicated. We also have a simple formula in the case that A has exactly one eigenvalue repeated n times.

Theorem 2. *Suppose that A is an $n \times n$ matrix with exactly one eigenvalue λ with algebraic multiplicity n . Then*

$$\exp(At) = e^{\lambda t} \sum_{j=1}^n \frac{1}{j!} (A - \lambda I)^j t^j.$$

Fooling around with shortcuts like this can be a lot of fun. How about you try to use it to calculate the matrix exponential of some interesting matrices?