Math 309 Section F
Fall 2015
Final
December 17, 2015
Time Limit: 1 Hour 50 Minutes

Name (Print): ________________________

Student ID: ________________________

This exam contains 10 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	15	
8	15	
9	10	
Total:	100	

1. (10 points) Find the general solution of the equation

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 21 & -10 \\ 36 & -17 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$$

2. (a) (5 points) Find the solution of the heat equation problem

$$u_t - 4u_{xx} = 0, \quad u(0,t) = 0, \ u(4,t) = 0, \quad u(x,0) = \sin(3\pi x) - 2\sin(3\pi x/2).$$

(b) (5 points) Find the solution of the heat equation problem

$$u_t - 7u_{xx} = 0, \quad u_x(0,t) = 0, \ u_x(15,t) = 0, \quad u(x,0) = 1 - \cos(\pi x/3) + 8\cos(3\pi x/5).$$

3. (10 points) Find the $2L$-periodic sine transform of the function $f(x)$ defined on the interval $[0, L]$ by

$$f(x) = \begin{cases} 2x/L, & 0 \le x \le L/2 \\ 2 - 2x/L, & L/2 \le x \le L \end{cases} .$$

4. (10 points) Consider the wave equation problem

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(0,t) = 0, u(L,t) = 0,\ t > 0$$

$$u(x,0) = f(x), u_t(x,0) = g(x),\ 0 \le x \le L$$

(a) Find the Fourier series solution $u(x,t)$ of the above wave equation for $0 \le x \le L$ and $t > 0$ (eg. do not use D'Alembert), in the case when $g(x) = 0$ and $f(x)$ is defined on the interval $[0, L]$ by

$$f(x) = \begin{cases} 2x/L, & 0 \le x \le L/2 \\ 2 - 2x/L, & L/2 \le x \le L \end{cases} .$$

(b) Find the Fourier series solution $u(x,t)$ of the above wave equation for $0 \le x \le L$ and $t > 0$ (eg. do not use D'Alembert), in the case when $f(x) = 0$ and $g(x)$ is defined on the interval $[0, L]$ by

$$g(x) = \begin{cases} 2x/L, & 0 \le x \le L/2 \\ 2 - 2x/L, & L/2 \le x \le L \end{cases} .$$

5. (10 points) Consider solutions to Laplace's equation $u_{xx} + u_{yy} = 0$ in the rectangle bounded by the lines $x = 0, x = 2, y = 0$, and $y = 3$.

   (a) Using separation of variables, find a fundamental set of solutions in the rectangle satisfying homogeneous boundary conditions on the lines $x = 0, x = 2$, and $y = 0$, e.g.

   $$u(0, y) = 0, \;\; u(2, y) = 0, \;\; u(x, 0) = 0,$$

   for $0 \leq y \leq 3$ and $0 \leq x \leq 2$. [Note: there should be one solution (up to an arbitrary constant) for every positive integer n].

   (b) Using the solutions you found in (a), find a solution to Laplace's equation in the rectangle satisfying the fourth boundary condition:

   $$u(x, 3) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$$

   for $0 \leq x \leq 2$. [Note: this last function is the same as in Problem 2, with $L = 2$]

6. **TRUE or FALSE section**. For each of the following statements, write TRUE if the statement is true and FALSE if the statement is false, in big uppercase letters. Expressions of the form FAUE, TRALSE, etc. will recieve no credit.

   (a) (2 points) Any function may be written as the sum of an even function and an odd function.

   (b) (2 points) Every solution to the one-dimensional heat equation is of the form $u(x,t) = F(x)G(t)$ for some function $F(x)$ and some function $G(t)$.

   (c) (2 points) If $u(x,y)$ and $v(x,y)$ are two solutions to the wave equation, both satisfying the same initial condition and the same boundary conditions, then $u(x,y)$ and $v(x,y)$ are equal.

   (d) (2 points) A symmetric matrix is always diagonalizable.

   (e) (2 points) If A is a 2×2 matrix with a negative eigenvalue, then $\binom{0}{0}$ is an asymptotically stable equilibrium of the system $\binom{x'}{y'} = A\binom{x}{y}$.

7. (15 points) Calculate the Fourier series of the function

$$f(x) = \begin{cases} -x/2, & -2 \le x < 0 \\ 2x - x^2/2, & 0 \le x < 2, \end{cases}$$

with $f(x+4) = f(x)$ for all x.

8. (15 points) Find the general solution of the system of equations

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t), \quad A = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}.$$

9. (10 points) The radially symmetric heat equation in spherical coordinates is given by

$$u_t - \frac{2}{r}u_r - u_{rr} = 0.$$

(a) Use separation of variables to reduce this to a system of two ordinary differential equations

(b) Show that $u(r, t) = A/r + B$ is a solution to the above equation