**Math 309 Section C**
**Spring 2016**
**Midterm**
**May 4, 2016**
**Time Limit: 50 Minutes**

**Name (Print):** ____________________

**Student ID:** ____________________

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This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 50 | |

1. (a) (5 points) Find a solution to the initial value problem

$$x'(t) = x(t) + 4y(t)$$
$$y'(t) = -10x(t) + 5y(t)$$

satisfying the initial condition $x(0) = 1, y(0) = 0$.

(b) (5 points) Find a particular solution of the equation for $t > 0$

$$x'(t) = x(t) + 2y(t) - 2$$
$$y'(t) = 2x(t) - y(t) + t$$

2. (10 points) In his latest ploy for world domination, Lex Luthor has installed a device which allows him to control the current at the surface of a strategically controlled patch of ocean. The device has a tuning parameter $c$ which Luthor may set to various real values in order to alter the resulting behavior of the system. The device is such that (ignoring inertia) a boat in the ocean will satisfy the differential equation

$$\frac{d}{dt}\vec{y} = A\vec{y}, \quad A = \begin{bmatrix} c & -1 \\ 2c & -1 \end{bmatrix}.$$

Qualitatively describe the behavior of a boat starting near (but not at) the position $\binom{0}{0}$ for all of the possible values of $c$. In other words, classify the critical point at the origin.

3. For each of the following statements, write TRUE if the statement is TRUE, and FALSE if the statement is false. If the statement is false, **also provide a counter-example**.

   (a) (2 points) If $\Psi(x)$ and $\Phi(x)$ are both fundamental matrices for the differential equation $\frac{d}{dt}\vec{y} = A\vec{y}$, then $\Phi(x)^{-1}\Psi(x)$ is a constant matrix.

   (b) (2 points) Suppose $A$ is an $3 \times 3$ square matrix, and that $A$ has eigenvalues $\lambda_1$ and $\lambda_2$ which have algebraic multiplicity 2 and 1, respectively. Then $A$ is not diagonalizable.

   (c) (2 points) Suppose $A$ is an $n \times n$ nondegenerate matrix, and that $A$ has 1 as an eigenvalue with algebraic multiplicity $n$. Then $A = I$.

   (d) (2 points) If $A$ is a real matrix, then the eigenvalues of $A$ will also be real.

   (e) (2 points) If a square matrix $A^2$ is diagonalizable, then $A$ is diagonalizable.

4. For each of the following, give an example if an example exists. If an example does not exist, then write DOES NOT EXIST in big bold letters.

   (a) (2 points) An $n \times n$ matrix with no eigenvalues.

   (b) (2 points) Two different fundamental matrices for the equation $\frac{d}{dt}\vec{y} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{y}$

   (c) (2 points) Two different $2 \times 2$ matrices both of which have eigenvalue 42 with algebraic multiplicity 2.

   (d) (2 points) A collection of linearly independent functions whose Wronskian is zero.

   (e) (2 points) A nondegenerate square matrix $A$ such that $A^2$ is degenerate.

5. Consider the $3 \times 3$ matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) (3 points) Determine the eigenvalues of $A$ and their algebraic multiplicities

(b) (3 points) For each eigenvalue $\lambda$ of $A$, determine its geometric multiplicity and the eigenspace $E_\lambda(A)$

(c) (4 points) Find an invertible matrix $P$ and a matrix $N$ in Jordan normal form such that $P^{-1}AP = N$