Math 309 Section F
Fall 2015
Midterm
October 30, 2015
Time Limit: 50 Minutes

Name (Print): ______________________

Student ID: ______________________

This exam contains 14 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	15	
2	15	
3	15	
4	10	
5	10	
6	10	
Total:	75	

Do not write in the table to the right.

1. (15 points) As quickly as you can, write down a *real* fundamental matrix for each of the following systems of equations.

   (a)

$$\begin{aligned} x' &= x - 4y \\ y' &= 4x - 7y \end{aligned}$$

   (b)

$$\begin{aligned} x' &= x - y \\ y' &= 5x - 3y \end{aligned}$$

   (c)

$$\begin{aligned} x' &= -2x + y \\ y' &= x - 2y \end{aligned}$$

Solution 1.

1. We have that
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}.$$
The characteristic polynomial of A is $r^2 + 6r + 9$, and therefore the eigenvalues of A are $-3, -3$. A fundamental matrix is therefore
$$\Psi(t) = \exp(At) = (I + (A + 3I)t)e^{-3t} = \begin{pmatrix} e^{3t} + 4te^{3t} & -4te^{3t} \\ 4te^{3t} & e^{3t} - 4te^{3t} \end{pmatrix}$$

2. We have that
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}.$$
The characteristic polynomial of A is $r^2 + 2r + 2$, and the eigenvalues of A are therefore $-1 \pm i$. A fundamental matrix is therefore
$$\Psi(t) = \exp(At) = e^{-t}(\cos(t)I + \frac{1}{1}(A+I)\sin(t)) = \begin{pmatrix} e^{-t}\cos(t) + 2e^{t}\sin(t) & -e^{-t}\sin(t) \\ 5e^{-t}\sin(t) & e^{-t}\cos(t) - 2e^{-t}\sin(t) \end{pmatrix}.$$

3. We have that
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$
The characteristic polynomial of A is $r^2 + 4r + 3 = (r + 3)(r + 1)$. The eigenvalues of A are therefore $-1, -3$. A fundamental matrix is therefore
$$\Psi(t) = \exp(At) = \frac{1}{-2}e^{-3t}(A+I) + \frac{1}{2}e^{-t}(A+3I) = \begin{pmatrix} (1/2)e^{-3t} + (1/2)e^{-t} & (-1/2)e^{-3t} + (1/2)e^{-t} \\ (-1/2)e^{-3t} + (1/2)e^{-t} & (1/2)e^{-3t} + (1/2)e^{-t} \end{pmatrix}$$

ALTERNATIVE SOLUTION FOR (c): Alternatively, we can determine the eigenspaces for each eigenvalue of A:
$$E_{-1}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, E_{-3}(A) = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$$
This means that
$$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-3t}\right\}$$
forms a fundamental set of solutions for our linear system of differential equations. Therefore
$$\Phi(t) = \begin{pmatrix} e^{-t} & -e^{-3t} \\ e^{-t} & e^{3t} \end{pmatrix}$$
is a fundamental matrix. It's different from the previous fundamental matrix, but that's ok – there's more than one fundamental matrix!

2. (15 points) Werewolves and vampires do not prey on one-another, but do complete for the same limited food supply – the resident human population. In this way, increases in the vampire population lead to decreases in the human population and vise-versa. Let v and w be the populations of vampires and werewolves (in units of 1000) at time t. According to the empirical observations of Buffy the Vampire Slayer, the Winchester brothers, Bruce Campbell, Al Gore, and other experts, the qualitative behavior of the populations of each monster species is governed by the equations

$$\begin{aligned} v'(t) &= v(1 - v - w) \\ w'(t) &= w(0.75 - w - 0.5v) \end{aligned}$$

(a) Determine the critical points of the above system.

(b) 0ne of the critical points you found in (a) should have been $(0.5, 0.5)$. Determine whether this critical point is stable, asymptotically stable, or unstable, and also whether it is a node, saddle, or spiral point. [Hint: you may want to use the Jacobian]

(c) Suppose that the initial population consists of 250 vampires and 500 werewolves. Qualitatively describe the populations at large time t

Solution 2.

(a) Critical points occur at values of v and w satisfying

$$\begin{cases} v(1-v-w)=0 \\ w(0.75-w-0.5v)=0 \end{cases}$$

Looking at the first equation either $v=0$ or $1-v-w=0$. Let's first assume that $v=0$. If $v=0$, then the second equation says that $w(0.75-w)=0$ and therefore either $w=0$ or $w=0.75$. Thus we've already found two critical points: $(0,0)$ and $(0,0.75)$. Let's assume instead that $v \neq 0$. Then the first equation says that $(1-v-w)=0$, and therefore that $v=1-w$. Putting this into the second equation, we obtain $w(0.75-w-0.5(1-w))=0$, which simplifies to $w(0.25-0.5w)=0$. Therefore either $w=0$ or $w=0.5$. If $w=0$, then since $v=1-w$ we would have $v=1$ giving us the critical point $(1,0)$. If $w=0.5$, then since $v=1-w$, we would have $v=0.5$, giving us the critical point $(0.5,0.5)$. To summarize, we've found four critical points:

$$\{(0,0),(0,0.75),(1,0),(0.5,0.5)\}$$

(b) We first calculate the Jacobian:

$$J(v,w)=\begin{pmatrix} \frac{\partial}{\partial v}(v(1-v-w)) & \frac{\partial}{\partial w}(v(1-v-w)) \\ \frac{\partial}{\partial v}(w(0.75-w-0.5v)) & \frac{\partial}{\partial w}(w(0.75-w-0.5v)) \end{pmatrix} = \begin{pmatrix} 1-2v-w & -v \\ -0.5w & 0.75-2w-0.5v \end{pmatrix}$$

Evaluating this at the point of interest $(v,w)=(0.5,0.5)$, we obtain the matrix

$$J(0.5,0.5)=\begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix}$$

The characteristic polynomial of this matrix is $r^2+r+(1/8)$, and the eigenvalues are therefore $-(1/2)\pm\sqrt{3}/4$, which are both negative. This means that the critical point at $(0.5,0.5)$ is an asymptotically stable node point.

(c) Since $(0.5,0.5)$ is an asymptotically stable node point, at large times the populations should be very close to $(0.5,0.5)$. In other words, at very large times, the populations should be very close to 500 vampires and 500 werewolves.

3. (15 points) Consider the matrix

$$A = \begin{pmatrix} 1 & -4 & -4 \\ -2 & 3 & 4 \\ 2 & -4 & -5 \end{pmatrix}$$

(a) determine the eigenvalues of A

(b) for each eigenvalue, calculate the corresponding eigenspace and determine its dimension. Is the matrix diagonalizable?

(c) calculate the value of A^{2015}

Solution 3.

(a) To do this we need to calculate the determinant of $A - xI$. We can simply do this "brute force":

$$\begin{aligned}\det(A - xI) &= \det\begin{pmatrix} 1-x & -4 & -4 \\ -2 & 3-x & 4 \\ 2 & -4 & -5-x \end{pmatrix} \\ &= (1-x)[(3-x)(-5-x)+16] + 4[-2(-5-x)-8] - 4[8-(3-x)2] \\ &= (1-x)[(3-x)(-5-x)+16] = (1-x)(1+2x+x^2) = (1-x)(1+x)^2\end{aligned}$$

Alternatively, we can use a bit of trickery! One very useful fact is that adding a multiple of one row to another, or a multiple of one column to another, doesn't change the determinant! Therefore

$$\begin{aligned}\det(A - xI) &= \det\begin{pmatrix} 1-x & -4 & -4 \\ -2 & 3-x & 4 \\ 2 & -4 & -5-x \end{pmatrix} = \det\begin{pmatrix} 1-x & -4 & -4 \\ -2 & 3-x & 4 \\ 0 & -1-x & -1-x \end{pmatrix} \\ &= \det\begin{pmatrix} 1-x & 0 & -4 \\ -2 & -1-x & 4 \\ 0 & 0 & -1-x \end{pmatrix} = \det\begin{pmatrix} 1-x & 0 & -4 \\ 0 & -1-x & 4 \\ 0 & 0 & -1-x \end{pmatrix} \\ &= (1-x)(-1-x)(-1-x) = (1-x)(1+x)^2\end{aligned}$$

In any event, the eigenvalues are 1 with algebraic multiplicity one and -1 with algebraic multiplicity two.

(b) We calculate via row reduction

$$E_1(A) = \mathcal{N}(A - I) = \mathcal{N}\begin{pmatrix} 0 & -4 & -4 \\ -2 & 2 & 4 \\ 2 & -4 & -6 \end{pmatrix} = \mathcal{N}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$$E_{-1}(A) = \mathcal{N}(A - I) = \mathcal{N}\begin{pmatrix} 2 & -4 & -4 \\ -2 & 4 & 4 \\ 2 & -4 & -4 \end{pmatrix} = \mathcal{N}\begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right\}$$

This shows that the geometric multiplicity of 1 is one and the geometric multiplicity of -1 is two. Therefore for every eigenvalue of A, the algebraic and geometric multiplicities are in agreement, and this means that A is diagonalizable.

(c) There were a lot of "close" solutions that people found for (c), so we will present a couple.
Solution 1:
The matrix A is diagonalizable with eigenvalues ± 1, and therefore the matrix A^2 is also diagonalizable with all of its eigenvalues equal to 1. This means that $A^2 = I$ (this could also be shown explicitly). Therefore

$$A^{2015} = A^{2014}A = (A^2)^{1007}A = I^{1007}A = IA = A.$$

Solution 2:

The matrix A is diagonalizable, and from (b) we know that

$$P^{-1}AP = D, \quad P = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This means that $A = PDP^{-1}$ and therefore

$$A^{2015} = (PDP^{-1})^{2015} = PD^{2015}P^{-1} = P \begin{pmatrix} 1^{2015} & 0 & 0 \\ 0 & (-1)^{2015} & 0 \\ 0 & 0 & (-1)^{2015} \end{pmatrix} P^{-1} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} P^{-1}$$

In either case, it turns out A^{2015} is just A again!

4. Provide an example of each of the following, if an example exists. If no example exists, write DOES NOT EXIST in big, bold text.

   (a) (2 points) Two *different* 3×3 matrices A and B, each with eigenvalues $1, -2, 4$

   (b) (2 points) A 3×3 matrix with a generalized eigenvector of rank 3 (just the matrix; you don't need to tell me a vector)

   (c) (2 points) A square matrix A whose matrix exponential $\exp(A)$ is not invertible

   (d) (2 points) A two-dimensional subspace of $\mathbb{R}^4$

   (e) (2 points) A linear system of algebraic equations with exactly two solutions.

Solution 4.

(a)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, B = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(b) Just take any 3×3 Jordan block:

$$J_3(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

(c) This does not exist, since $\exp(A)$ is always invertible – it's inverse is $\exp(-A)$

(d) Just take the span of two linearly independent vectors in $\mathbb{R}^4$, eg

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(e) This does not exist, since a linear system of algebraic equations always has no solutions, a unique solution, or infinitely many solutions.

5. (10 points) Find a solution to the differential equation

$$\frac{d}{dt}\vec{y}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{y}(t).$$

satisfying the initial condition $x(0) = 3, y(0) = -1$.

Solution 5. The first thing that we need to do is find the general solution. One nice way to do this is by calculating a fundamental matrix. The differential equation we are trying to solve is

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t), \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

The characteristic polynomial of A is $r^2 - 2r - 3 = (r-3)(r+1)$. Therefore the eigenvalues are $3, -1$. Thus

$$\Psi(t) = \exp(At) = \frac{1}{4}e^{3t}(A+I) - \frac{1}{4}e^{-t}(A-3I) = \frac{1}{4}\begin{pmatrix} 2e^{3t}+2e^{-t} & e^{3t}-e^{-t} \\ 4e^{3t}-4e^{-t} & 2e^{3t}+2e^{-t} \end{pmatrix}.$$

The general solution is then

$$\vec{y}(t) = \Psi(t)\vec{c} = \frac{1}{4}\begin{pmatrix} 2e^{3t}+2e^{-t} & e^{3t}-e^{-t} \\ 4e^{3t}-4e^{-t} & 2e^{3t}+2e^{-t} \end{pmatrix}\vec{c}$$

for some arbitrary constant vector $\vec{c}$. Lastly, we need to figure out what $\vec{c}$ needs to be for our solution to satisfy the initial condition specified. Note that since we chose the fundamental matrix $\Psi(t)$ to be the matrix exponential, we know that $\Psi(0) = I$ and therefore

$$\vec{y}(0) = \Psi(0)\vec{c} = I\vec{c} = \vec{c}$$

since $\vec{y}(0) = \binom{3}{-1}$, this means that $\vec{c} = \binom{3}{-1}$ and therefore

$$\vec{y}(t) = \Psi(t)\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5e^{3t}+7e^{-t} \\ 10e^{3t}-14e^{-t} \end{pmatrix}.$$

6. (10 points) Find the general solution of the equation

$$\frac{d}{dt}\vec{y}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{y}(t) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$$

Solution 6. Since we already know the general solution $\vec{y}_h(t)$ of the corresponding homogeneous equation from the previous problem, we just need to find a particular solution $\vec{y}_p(t)$ to the nonhomogeneous equation and throw everything together to get the general solution to the nonhomogeneous equation. We will present two different solutions to accomplish this– out of the many different ways that we could use.

SOLUTION 1: We may use the *method of undetermined coefficients* to try to find a solution. Just like in Math 307, we propose a solution of the form $\vec{y}_p(t) = \binom{a}{b}e^t$. Then $\frac{d}{dt}\vec{y}_p(t) = \binom{a}{b}e^t$ and putting this into the nonhomogeneous differential equation we find

$$\begin{pmatrix} a \\ b \end{pmatrix} e^t = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} e^t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$$

Dividing out by e^t, this says

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

This simplifies to

$$\begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Solving this linear system of equations, we fund $a = 1/4$ and $b = -2$, and therefore $\vec{y}_p = \binom{1/4}{-2}e^t$ is a particular solution. The general solution is therefore

$$\vec{y}(t) = \vec{y}_p(t) + \vec{y}_h(t) = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t + \frac{1}{4} \begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix} \vec{c}.$$

SOLUTION 2: We may use the *method of variation of parameters* to try to find a solution. We use the fundamental matrix $\Psi(t)$ from the previous problem, noting that since it is a matrix exponential, $\Psi(t)^{-1} = \Psi(-t)$. Therefore from variation of parameters, we know that the solution will be

$$\begin{aligned}
\vec{y}_p(t) &= \Psi(t) \int \Psi(t)^{-1} \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t dt = \Psi(t) \int \Psi(-t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t dt \\
&= \Psi(t) \int \frac{1}{4} \begin{pmatrix} 2e^{-3t} + 2e^t & e^{-3t} - e^t \\ 4e^{-3t} - 4e^t & 2e^{-3t} + 2e^t \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t dt \\
&= \Psi(t) \int \frac{1}{4} \begin{pmatrix} 3e^{-2t} + 5e^{2t} \\ 6e^{-2t} - 10e^{2t} \end{pmatrix} dt = \Psi(t) \frac{1}{8} \begin{pmatrix} -3e^{-2t} + 5e^{2t} \\ -6e^{-2t} - 10e^{2t} \end{pmatrix} \\
&= \frac{1}{32} \begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix} \begin{pmatrix} -3e^{-2t} + 5e^{2t} \\ -6e^{-2t} - 10e^{2t} \end{pmatrix} = \begin{pmatrix} 1/4e^t \\ -2e^t \end{pmatrix}
\end{aligned}$$

The general solution is therefore

$$\vec{y}(t) = \vec{y}_p(t) + \vec{y}_h(t) = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t + \frac{1}{4} \begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix} \vec{c}.$$