MATH 309: Homework #2

Due on: April 15, 2016

Problem 1 *Jordan Normal Form*

For each of the following values of the matrix A, find an invertible matrix P and a matrix N in Jordan normal form such that $P^{-1}AP = N$.

(a)
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(b)
$$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

(c)
$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(d)
$$A = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

(e)
$$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

(f)
$$A = \begin{pmatrix} 0 & 0 & 24 \\ 1 & 0 & 2 \\ 0 & 1 & -5 \end{pmatrix}$$

(g)
$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{pmatrix}$$

(h)
$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

(i)
$$A = \begin{pmatrix} -1 & -1 & 0 \\ 4 & 3 & 0 \\ -6 & -3 & 1 \end{pmatrix}$$

. . . .

Solution 1.

(a) The matrix A is already in Jordan normal form, so we can take $P = I$ and $N = A$.

(b) The characteristic polynomial is $p_A(x) = x^2 - 3x + 3$. The eigenvalues are therefore $(3/2) \pm i\sqrt{3}/2$. Therefore the matrix is diagonalizable, eg. its Jordan normal form N will be a diagonal matrix. Finding an eigenvector for each eigenvalue, we get

$$P = \begin{pmatrix} (-1/2) + i\sqrt{3}/2 & (-1/2) - i\sqrt{3}/2 \\ 1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} (3/2) + i\sqrt{3}/2 & 0 \\ 0 & (3/2) - i\sqrt{3}/2 \end{pmatrix}$$

(c) The characteristic polynomial is $p_A(x) = x^2-2x+2$. The eigenvalues are therefore $1 \pm i$. Therefore the matrix is diagonalizable, eg. its Jordan normal form N will be a diagonal matrix. Finding an eigenvector for each eigenvalue, we get

$$P = \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

(d) The charcteristic polynomial is $p_A(x) = x^2+2x-1$. The eigenvalues are therefore $-1 \pm \sqrt{2}$. Therefore the matrix is diagonalizable, eg. its Jordan normal form N will be a diagonal matrix. Finding an eigenvector for each eigenvalue, we get

$$P = \begin{pmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} -1+\sqrt{2} & 0 \\ 0 & -1-\sqrt{2} \end{pmatrix}$$

(e) The characteristic polynomial is $p_A(x) = x^2 + 2x + 1$. The eigenvalue is -1 with algebraic multiplicity two. However the geometric multiplicity is one and so N will be a 2×2 Jordan block $N = J_2(-1)$. Note that *any* nonzero vector in $\mathbb{R}^2$ will be a generalized eigenvector, since $(A - (-1)I)^2 = 0$. Note that $\vec{v} = \binom{1}{0}$ is not an eigenvector of A, and so it must be a generalized eigenvector of rank 2. This means that

$$\vec{w} := (A - (-1)I)\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector of A with eigenvalue -1. Thus we may take

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad N = J_2(-1) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

(f) The characteristic polynomial of this matrix is $p_A(x) = -x^3 - 5x^2 + 2x + 24 = -(x-2)(x+3)(x+4)$. The eigenvalues are therefore $2, -3, -4$. The matrix is therefore diagonalizable, eg. its Jordan form N will be a diagonal matrix. Finding an eigenvector for each eigenvalue, we get

$$P = \begin{pmatrix} 12 & -8 & -6 \\ 7 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, N = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

(g) The characteristic polynomial of this matrix is $p_A(x) = -x^3 - 3x^2 - 3x - 1 = -(x+1)^3$. The eigenvalue is therefore -1 with algebraic multiplicity 3. We calculate the corresponding eigenspace

$$E_{-1}(A) = N(A - (-1)I)) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

Therefore the geometric multiplicity is 1. It follows that the Jordan normal form of A is a 3×3 Jordan block $N = J_3(-1)$. We find a generalized eigenvector of rank 2 by solving

$$(A - (-1)I)\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

A solution is

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We also need a generalized eigenvector of rank 3 in this case, which we obtain by solving the equation

$$(A - (-1)I)\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

A solution is

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Putting this all together, we have

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, N = J_3(-1) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

(h) The characteristic polynomial is $p_A(x) = -x^3 + 3x - 2 = (x-1)^2(x+2)$. We calculate the corresponding eigenspaces

$$E_{-2}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$E_1(A) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\}$$

In particular, this shows that 1 has algebraic multiplicity two but geometric multiplicity 1, and so we must still find a generalized eigenvector of rank two with eigenvalue 1. We can do this by solving the equation

$$(A - (1)I)\vec{v} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

A solution is given by

$$\vec{v} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}.$$

Putting this all together we have

$$P = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 1 & -1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) The characteristic polynomial of A is $p_A(x) = -(x-1)^3$, and therefore we have the eigenvalue 1 with algebraic multiplicity three. Furthermore, we calculate the eigenspace

$$E_1(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Therefore the geometric multiplicity of 1 is two. This means that we need to find a generalized eigenvector of rank 2. How can we do this? We can try to solve

$$(A-(1)I)\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

but we find that this has *no solution*. Similarly can try to solve

$$(A-(1)I)\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

but again this has *no solution*. What's the deal? The answer is we need to choose the "right" basis for the eigenspace $E_1(A)$. The way to do this is to find a generalized eigenvector first, and then find the regular eigenvectors after. Finding a generalized eigenvector is easy, actually. One may check that $(A-(1)I)^2 = 0$, and therefore every nonzero vector in $\mathbb{R}^3$ is a generalized eigenvector of A with eigenvalue λ. Choose any one that is not already an eigenvector of A, say

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then we know that $\vec{v}$ is a generalized eigenvector of A, and therefore must have rank 2. Now if we define $\vec{w}$ by

$$\vec{w} := (A-(1)I)\vec{v} = \begin{pmatrix} -2 \\ 4 \\ -6 \end{pmatrix}$$

then $\vec{w}$ is an eigenvector of A with eigenvalue 1. Finally if we choose any other eigenvector $\vec{e}$ to complete a basis for $E_1(A)$ (e.g. $\vec{e} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ but it doesn't matter which), we have the three vectors which will work as the column vectors for P:

$$P = (\vec{e}\ \vec{v}\ \vec{w}) = \begin{pmatrix} -2 & 1 & 0 \\ 4 & 0 & 0 \\ -6 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note here that the positioning of the column vectors in P is delicate and important. The generalized eigenvectors corresponding to a particular Jordan block need to be positioned in order of increasing rank.

Problem 2 Matrix Exponential

For each of the values of the matrix A in the previous problem, determine the value of $\exp(At)$

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Solution 2.

(a) We have the same eigenvalue 1, repeated twice, so

$$\exp(At) = e^t(I + (A - I)t) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

(b) We have two complex eigenvalues $a \pm ib$ for $a = 3/2$ and $b = \sqrt{3}/2$, and therefore

$$\begin{aligned} \exp(At) &= e^{(3/2)t}\cos((\sqrt{3}/2)t)I + e^{(3/2)t}\frac{2}{\sqrt{3}}(A - (3/2)I)\sin((\sqrt{3}/2)t) \\ &= \begin{pmatrix} e^{(3/2)t}\cos((\sqrt{3}/2)t) + (-1/\sqrt{3})e^{(3/2)t}\sin((\sqrt{3}/2)t) & (-5/\sqrt{3})e^{(3/2)t}\sin((\sqrt{3}/2)t) \\ (-1/\sqrt{3})e^{(3/2)t}\sin((\sqrt{3}/2)t) & e^{(3/2)t}\cos((\sqrt{3}/2)t) + (1/\sqrt{3})e^{(3/2)t}\sin((\sqrt{3}/2)t) \end{pmatrix} \end{aligned}$$

(c) We have two complex eigenvalues $a \pm ib$ for $a = 1$ and $b = 1$, and therefore

$$\exp(At) = e^t\cos(t)I + e^t(A - (1)I)\sin(t) = \begin{pmatrix} e^t\cos(t) & e^t\sin(t) \\ -e^t\sin(t) & e^t\cos(t) \end{pmatrix}$$

(d) The eigenvalues are real and distinct, given by $r_1 = -1 + \sqrt{2}$ and $r_2 = -1 - \sqrt{2}$ and therefore

$$\begin{aligned} \exp(At) &= \frac{1}{2\sqrt{2}}e^{(-1+\sqrt{2})t}(A - (-1 - \sqrt{2})I) - \frac{1}{2\sqrt{2}}e^{(-1-\sqrt{2})t}(A - (-1 + \sqrt{2})I) \\ &= \frac{1}{2\sqrt{2}}\begin{pmatrix} (1 + \sqrt{2})e^{(-1+\sqrt{2})t} - (1 - \sqrt{2})e^{(-1-\sqrt{2})t} & e^{(-1+\sqrt{2})t} - e^{(-1-\sqrt{2})t} \\ e^{(-1+\sqrt{2})t} - e^{(-1-\sqrt{2})t} & (-1 + \sqrt{2})e^{(-1+\sqrt{2})t} - (-1 - \sqrt{2})e^{(-1-\sqrt{2})t} \end{pmatrix} \end{aligned}$$

(e) The eigenvalue of A is -1 repeated twice. Therefore

$$\exp(At) = e^{-t}(I + (A - (-1)I)t) = \begin{pmatrix} e^{-t} + te^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{pmatrix}$$

(f) The eigenvalues of A are $r_1 = 2, r_2 = -3, r_3 = -4$. These are all distinct, so by Sylvester's formula we have that

$$\begin{aligned}\exp(At) &= e^{r_1 t}\frac{1}{(r_1 - r_2)(r_1 - r_3)}(A - r_2 I)(A - r_3 I)\\ &+ e^{r_2 t}\frac{1}{(r_2 - r_1)(r_2 - r_3)}(A - r_1 I)(A - r_3 I)\\ &+ e^{r_3 t}\frac{1}{(r_3 - r_1)(r_3 - r_2)}(A - r_1 I)(A - r_2 I)\\ &= \frac{1}{30}e^{2t}(A + 3I)(A + 4I) - \frac{1}{5}e^{-3t}(A - 2I)(A + 4I) + e^{-4t}\frac{1}{6}(A - 2I)(A + 3I)\end{aligned}$$

Calculating this we get the matrix

$$\left(\begin{matrix} (12/30)e^{2t} + (8/5)e^{-3t} - (6/6)e^{-4t} & (24/30)e^{2t} - (24/5)e^{-3t} + (24/6)e^{-3t} & (48/30)e^{2t} + (72/5)e^{-3t} - (96/6)e^{-} \\ (7/30)e^{2t} - (2/5)e^{-3t} + (1/6)e^{-4t} & (14/30)e^{2t} + (6/5)e^{-3t} - (4/6)e^{-3t} & (28/30)e^{2t} - (18/5)e^{-3t} + (16/6)e^{-} \\ (1/30)e^{2t} - (1/5)e^{-3t} + (1/6)e^{-4t} & (2/30)e^{2t} + (3/5)e^{-3t} - (4/6)e^{-3t} & (4/30)e^{2t} - (9/5)e^{-3t} + (16/6)e^{-3t} \end{matrix}\right.$$

(g) The eigenvalues of this matrix are all -1 (repeated three times). Therefore

$$\exp(At) = e^{-t}(I + (A - (-1)I)t + \frac{1}{2}(A - (-1)I)^2 t^2) = e^{-t}\begin{pmatrix} 1 + t + \frac{1}{2}t^2 & -\frac{1}{2}t^2 & -t + \frac{1}{2}t^2 \\ t + t^2 & 1 + t - t^2 & -3t + t^2 \\ \frac{1}{2}t^2 & t - \frac{1}{2}t^2 & 1 - 2t + \frac{1}{2}t^2 \end{pmatrix}$$

(h) The eigenvalues of this matrix are -2, and 1 twice repeated. We calculate the matrix exponential using the Jordan normal form found in Problem 1 part (h). To remind ourselves, $P^{-1}AP = N$ with

$$P = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 1 & -1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

From this we have

$$\exp(At) = P\exp(Nt)P^{-1}$$

where

$$\exp(Nt) = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}$$

Therefore since

$$P^{-1} = \begin{pmatrix} 1/9 & -2/9 & 4/9 \\ 4/9 & 1/9 & -2/9 \\ -3/9 & -3/9 & -3/9 \end{pmatrix}$$

the answer is $P\exp(Nt)P^{-1}$, the calculation of which we leave to the reader.

(i) In this case the eigenvalues of A are 1 (repeated three times). Therefore

$$\exp(At) = e^{-t}(I + (A - (1)I)t + \frac{1}{2}(A - (1)I)^2 t^2) = e^t \begin{pmatrix} 1 - 2t & -t & 0 \\ 4t & 1 + 2t & 0 \\ -6t & -3t & 1 \end{pmatrix}$$

Problem 3 Fundamental Matrix

Find a fundamental matrix for each of the following systems of equations

(a)

$$\begin{aligned} x' &= x + y \\ y' &= x - y \end{aligned}$$

(b)

$$\begin{aligned} x' &= -x - 4y \\ y' &= x - y \end{aligned}$$

(c)

$$\begin{aligned} x' &= x + y \\ y' &= 4x - 2y \end{aligned}$$

(d)

$$\begin{aligned} x' &= -x - 4y \\ y' &= x - y \end{aligned}$$

(e)

$$\begin{aligned} x' &= x - y \\ y' &= 5x - 3y \end{aligned}$$

(f)

$$\begin{aligned} x' &= 3x - 4y \\ y' &= x - y \end{aligned}$$

(g)

$$\begin{aligned} x' &= 4x - 8y \\ y' &= 8x - 4y \end{aligned}$$

. . . .

Solution 3.

(a) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are $\pm\sqrt{2}$. Therefore we get that

$$\Psi(t) = \exp(At) = \frac{1}{2\sqrt{2}} \begin{pmatrix} (1+\sqrt{2})e^{\sqrt{2}t} - (1-\sqrt{2})e^{-\sqrt{2}t} & e^{\sqrt{2}t} - e^{-\sqrt{2}t} \\ e^{\sqrt{2}t} - e^{-\sqrt{2}t} & (-1+\sqrt{2})e^{\sqrt{2}t} - (-1-\sqrt{2})e^{-\sqrt{2}t} \end{pmatrix}$$

(b) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are $-1 \pm 2i$. Therefore we get that

$$\Psi(t) = \exp(At) = \begin{pmatrix} e^{-t}\cos(2t) & -2e^{-t}\sin(2t) \\ (1/2)e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{pmatrix}$$

(c) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are 2 and -3. Therefore we get that

$$\Psi(t) = \exp(At) = \frac{1}{5} \begin{pmatrix} 4e^{2t} + e^{-3t} & e^{2t} - e^{-3t} \\ 4e^{2t} - 4e^{-3t} & e^{2t} + 4e^{-3t} \end{pmatrix}$$

(d) This is a repeat of (b)

(e) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are $-1 \pm i$. Therefore we get that

$$\Psi(t) = \exp(At) = \begin{pmatrix} e^{-t}\cos(t) + 2e^{-t}\sin(t) & -e^{-t}\sin(t) \\ 5e^{-t}\sin(t) & e^{-t}\cos(t) - 2e^{-t}\sin(t) \end{pmatrix}$$

(f) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are 1, repeated twice. Therefore

$$\Psi(t) = \exp(At) = e^t(I + (A - I)t) = e^t \begin{pmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t - 2te^t \end{pmatrix}$$

(g) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are $\pm 4\sqrt{3}$. Therefore

$$\Psi(t) = \exp(At) = \frac{1}{8\sqrt{3}} \begin{pmatrix} (4 + 4\sqrt{3})e^{4\sqrt{3}t} - (4 - 4\sqrt{3})e^{-4\sqrt{3}t} & -8e^{4\sqrt{3}t} + 8e^{-4\sqrt{3}t} \\ 8e^{4\sqrt{3}t} - 8e^{-4\sqrt{3}t} & (-4 + 4\sqrt{3})e^{4\sqrt{3}t} - (-4 - 4\sqrt{3})e^{-4\sqrt{3}t} \end{pmatrix}$$

Problem 4 Matrix Sine and Cosine

NOT GRADED; DO NOT NEED TO DO

Let A be an $n \times n$ matrix. This problem concerns the matrix valued functions $\sin(At)$ and $\cos(At)$.

(a) Show that $\frac{d}{dt}\sin(At) = A\cos(At)$

(b) Show that $\frac{d}{dt}\cos(At) = -A\sin(At)$

(c) Let $\vec{v}, \vec{w} \in \mathbb{R}^n$. Show that

$$\vec{y}(t) := \cos(At) \cdot \vec{v} + \sin(At)\vec{w}$$

is a solution to the differential equation

$$\vec{y}''(t) = -A^2\vec{y}(t).$$

. . . .

Solution 4.

(a) We rely on the *definition* of matrix sine and matrix cosine. Recall that they are defined in terms of the Taylor series

$$\sin(At) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1} t^{2n+1} = At - \frac{1}{6}A^3t^3 + \frac{1}{5!}A^5t^5 - \frac{1}{7!}A^7t^7 + \dots$$

$$\cos(At) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} A^{2n} t^{2n} = I - \frac{1}{2}A^2t^2 + \frac{1}{4!}A^4t^4 - \frac{1}{6!}A^6t^6 + \dots$$

When we take the derivative:

$$\begin{aligned}\frac{d}{dt}\sin(At) &= A - \frac{3}{6}A^3t^2 + \frac{5}{5!}A^5t^4 - \frac{7}{7!}A^7t^6 + \dots \\ &= A - \frac{1}{2}A^3t^2 + \frac{1}{4!}A^5t^4 - \frac{1}{6!}A^7t^6 + \dots \\ &= A(I - \frac{1}{2}A^2t^2 + \frac{1}{4!}A^4t^4 - \frac{1}{6!}A^6t^6 + \dots) = A\cos(At).\end{aligned}$$

(b) When we take the derivative:

$$\begin{aligned}\frac{d}{dt}\cos(At) &= 0 - \frac{2}{2}A^2t + \frac{4}{4!}A^4t^3 - \frac{6}{6!}A^6t^5 + \dots \\ &= -A^2t + \frac{1}{3!}A^4t^3 - \frac{1}{5!}A^6t^5 + \dots \\ &= -A(At - \frac{1}{3!}A^3t^3 + \frac{1}{5!}A^5t^5 - \dots) = -A\sin(At).\end{aligned}$$

(c) Using (a) and (b), we calculate

$$\vec{y}'(t) = -A\sin(At)\cdot\vec{v} + A\cos(At)\vec{w}$$

and also

$$\vec{y}''(t) = -A^2\cos(At)\cdot\vec{v} - A^2\sin(At)\vec{w} = -A^2(\cos(At)\cdot\vec{v} + \sin(At)\cdot\vec{w}) = -A^2\vec{y}(t).$$

Therefore $\vec{y}(t)$ satisfies the differential equation.

Problem 5 Second-order differential equations

NOT GRADED; DO NOT NEED TO DO

Consider the differential equation

$$y''(t) + by'(t) + cy(t) = 0. \tag{1}$$

If we make the substitution, $z(t) = y'(t)$, then we may rewrite Equation (1) as a system of two first-order equations

$$\begin{cases} y'(t) &= z(t) \\ z'(t) &= -cy(t) - bz(t) \end{cases} \tag{2}$$

(a) Show that the characteristic polynomial of Equation (1) is the same as the characteristic polynomial of the matrix associated with the linear system in Equation (2).

(b) Find the fundamental matrix of the system in Equation (2) when $b = 5$ and $c = 4$

(c) Find the fundamental matrix of the system in Equation (2) when $b = 2$ and $c = 5$

(d) Find the fundamental matrix of the system in Equation (2) when $b = 2$ and $c = 1$.

(e) For (b)-(d), explain how the fundamental matrix you found corresponds to the general solution of Equation (1).

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Solution 5.

(a) The characteristic polynomial of Equation (1) is defined to be $x^2 + bx + c$. The matrix associated with Equation (2) is

$$A = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}$$

which has the characteristic polynomial $x^2 + bx + c$ (which is the same!).

(b) When $b = 5$ and $c = 4$, then the eigenvalues of A are -1 and -4, and therefore the fundamental matrix is

$$\Psi(t) = \exp(At) = \frac{1}{3}\begin{pmatrix} 4e^{-t} - e^{-4t} & e^{-t} - e^{-4t} \\ -4e^{-t} + 4e^{-4t} & -e^{-t} + 4e^{-4t} \end{pmatrix}$$

(c) When $b = 2$ and $c = 5$, then the eigenvalues of A are $-1 \pm 2i$, and therefore the fundamental matrix is

$$\Psi(t) = \exp(At) = \begin{pmatrix} e^{-t}\cos(t) + (1/2)e^{-t}\sin(t) & (1/2)e^{-t}\sin(t) \\ (-5/2)e^{-t}\sin(t) & e^{-t}\cos(t) - (1/2)e^{-t}\sin(t) \end{pmatrix}$$

(d) When $b = 2$ and $c = 1$, then the eigenvalues of A are -1 repeated twice, and therefore the fundamental matrix is

$$\Psi(t) = \exp(At) = \begin{pmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{pmatrix}$$

(e) In each case, the top two entries of the fundamental matrix form a fundamental set of solutions to Equation (1). In case (b), the two roots are real, and the fundamental set of solutions we get consist of exponentials with these coefficients. In case (c), the roots have an imaginary component which determines the frequency of the sine and cosine terms in the solution, just like back in Math 307. In the last case, we have a repeated root, and the solution that we get has $e^{\lambda t}$ and $te^{\lambda t}$ just like we learned to expect from our study of second-order differential equations.

Problem 6 A Zombie Outbreak

A zombie outbreak occurs in the isolated country Fictionland. Assume that the human per-capita birth rate in Fictionland is 0.013 and the per-capita death rate of humans is 0.008. The zombie outbreak leads to the conversion of humans to zombies at a rate of $0.003z(t)$, where $z(t)$ is the zombie population of Fictionland at time t. Humans also destroy the zombies at a rate of $dh(t)$, where $h(t)$ is the population of humans in Fictionland at time t. Assuming that at time $t = 0$, there is an equal population of humans and zombies. For which values of d does the human population eventually die out?

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Solution 6. To set up this problem, we must figure out differential equations describing the rate of change of the human and zombie populations as a function of time. The main idea is

$$\frac{dh}{dt} = (\text{rate in}) - (\text{rate out})$$

and similarly for $\frac{dz}{dt}$. What is the rate in for humans? The only way more humans appear is by being born. Therefore the rate in is $0.013h$. The way that humans leave the system is by dying or being zombified. Therefore the rate out is $0.008h + 0.003z$, and we may write

$$\frac{dh}{dt} = 0.013h - (0.008h + 0.003z) = 0.005h - 0.003z.$$

In a similar way, we find

$$\frac{dz}{dt} = -dh + 0.003z.$$

In terms of matrices, this says:

$$\begin{pmatrix} h' \\ z' \end{pmatrix} = A \begin{pmatrix} h \\ z \end{pmatrix}, \quad A = \begin{pmatrix} 0.005 & -0.003 \\ -d & 0.003 \end{pmatrix}.$$

Moreover, our initial condition is $h(0) = z(0) = c$ for some positive constant c. For what values of d will the human population never die out? Well, the human population will die out if $h(t)$ ever reaches 0. When can this happen?

Let's examine the eigenvalues of A. The characteristic polynomial of A is

$$p_A(x) = x^2 - 0.008x + 0.003(0.005 - d).$$

The roots are therefore

$$\frac{1}{2}\left(0.008 \pm \sqrt{0.008^2 - 4 * 0.003 * (0.005 - d)}\right)$$

Therefore if the quantity inside the radical is positive, then $\vec{0}$ is an unstable node or a saddle, and either way this forces the human population to shoot off toward infinity

– in particular the humans survive. If the quantity inside the radical is negative, then the origin is an unstable spiral. In this case, the solution traces a curve that rotates around the origin – in particular $h(t)$ will drop to zero at some point. The quantity in the radical is positive if and only if $d \geq -(1/3000)$, and for these values of d the human population survives. [The argument actually requires a little more elaboration, but this is good enough for full credit.]

Problem 7 Uniqueness of Fundamental Matrix

Let $A(t)$ be a matrix continuous on the interval (α, β). Show that if $\Psi(t)$ and $\Phi(t)$ are two fundamental matrices for the equation

$$\vec{y}'(t) = A(t)\vec{y}(t)$$

on the interval (α, β), then there exists a (constant) invertible matrix P so that $\Phi(t) = \Psi(t)P$.

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Solution 7. This is a tricky problem again – its solution will also be **extra credit**. Let $\Psi(t)$ and $\Phi(t)$ be two fundamental matrices for the equation, and choose $t_0 \in (\alpha, \beta)$. Set $P = \Psi(t_0)^{-1}\Phi(t_0)$. Then $\Phi(t_0) = \Psi(t_0)P$. Moreover, the column vectors of $\Phi(t)$ and $\Psi(t)$ are solutions to $\vec{y}'(t) = A(t)\vec{y}(t)$ (because they must form a fundamental set of solutions). The value of the first column of $\Phi(t)$ at $t = t_0$ agrees with the value of the first column of $\Psi(t)P$ at $t = t_0$. Therefore they both satisfy the same initial value problem, and by the Existence and Uniqueness Theorem this guarantees that they are equal for all t in the interval (α, β). The same argument in fact applies to the second column of each fundamental matrix, as well as the third, etc. Therefore $\Phi(t) = \Psi(t)P$ for all values of t.

Problem 8 Nonhomogeneous Equations

For each of the following, find the general solution.

(a)

$$\begin{cases} x' &= 2x - y + e^t \\ y' &= 3x - 2y + t \end{cases}$$

(b)

$$\begin{cases} x' &= x + y + e^{-2t} \\ y' &= 4x - 2y - 2e^t \end{cases}$$

(c)

$$\begin{cases} x' &= 2x - 5y - \cos(t) \\ y' &= x - 2y + \sin(t) \end{cases}$$

(d)

$$\begin{cases} x' &= -4x + 2y + t^3 \\ y' &= 2x - y - t^{-2} \end{cases}$$

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