MATH 309: Homework #3 Solutions

Due on: April 29, 2016

Problem 1 *A 2×2 Homogeneous Equation with Complex Eigenvalues*

Without using matrix exponentials, find a fundamental set of solutions for the system of equations

$$\frac{d}{dx}\vec{y} = A\vec{y}, \quad A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}.$$

[Remember: the real part and the imaginary part of a solution is also a solution!]

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Solution 1. The eigenvalues of the matrix are $3 \pm 2i$, and the corresponding eigenvectors are $\binom{\pm i}{-1}$. From this we have the two solutions

$$\vec{y}_1 = \begin{pmatrix} i \\ -1 \end{pmatrix} e^{(3+2i)x}, \qquad \vec{y}_2 = \begin{pmatrix} -i \\ -1 \end{pmatrix} e^{(3-2i)x}.$$

However, these are complex solutions. To get a real solution, we can choose one of them and takes its real and imaginary parts. Note that

$$\begin{aligned} \vec{y}_1 &= \begin{pmatrix} i \\ -1 \end{pmatrix} e^{(3+2i)x} = \begin{pmatrix} i \\ -1 \end{pmatrix} e^{3x} e^{2ix} \\ &= \begin{pmatrix} i \\ -1 \end{pmatrix} e^{3x}(\cos(2x) + i\sin(2x)) = \begin{pmatrix} -e^{3x}\sin(2x) \\ -e^{3x}\cos(2x) \end{pmatrix} + i \begin{pmatrix} e^{3x}\cos(2x) \\ -e^{3x}\sin(2x) \end{pmatrix} \end{aligned}$$

Therefore we have that

$$\mathrm{Re}(\vec{y}_1) = \begin{pmatrix} -e^{3x}\sin(2x) \\ -e^{3x}\cos(2x) \end{pmatrix}, \quad \mathrm{Im}(\vec{y}_1) = \begin{pmatrix} e^{3x}\cos(2x) \\ -e^{3x}\sin(2x) \end{pmatrix}$$

and these two real functions form a fundamental set of solutions.

Problem 2 *Stability of the Origin I*

Consider the matrix $A = \begin{pmatrix} c & 1 \\ 1 & 2 \end{pmatrix}$. For each value of c, classify the stability of the critical point at the origin for the equation

$$\frac{d}{dx}\vec{y} = A\vec{y}.$$

. . . .

Solution 2. We calculate $\mathrm{tr}(A) = c + 2$ and $\det(A) = 2c - 1$. As c ranges through all the real numbers, this parameterizes a straight line in the $\mathrm{tr}(A), \det(A)$-plane. The slope of the line is 2, and it passes through the point $\mathrm{tr}(A) = 0, \det(A) = -5$, so the equation of this line is

$$\det(A) = 2\mathrm{tr}(A) - 5.$$

This equation does not intersect with the critical curve $\det(A) = \frac{1}{4}\mathrm{tr}(A)^2$, and therefore the matrix A always has real values. Furthermore, by plotting the line in the $\mathrm{tr}(A), \det(A)$-plane we see that for our line of A's the origin is either a saddle point or an unstable node (ie source), and that the transition between these two occurs when our line crosses the $\mathrm{tr}(A)$-axis. This happens when $\det(A) = 0$, eg. $c = 1/2$, and therefore we have the following classification: the origin is a saddle point if $c < 1/2$, and an unstable node if $c > 1/2$.

Problem 3 *Stability of the Origin II*

Consider the matrix $A = \begin{pmatrix} c & 1 \\ -1 & 2 \end{pmatrix}$. For each value of c, classify the stability of the critical point at the origin for the equation

$$\frac{d}{dx}\vec{y} = A\vec{y}.$$

. . . .

Solution 3. We calculate $\mathrm{tr}(A) = c + 2$ and $\det(A) = 2c + 1$. Again, as c ranges through all real numbers, this parameterizes a straight line in the $\mathrm{tr}(A), \det(A)$-plane. The slope of the line is 2, and it passes through the point $\mathrm{tr}(A) = 0, \det(A) = -3$, so the equation of this line is

$$\det(A) = 2\mathrm{tr}(A) - 3.$$

This equation does intersect with the critical curve $\det(A) = \frac{1}{4}\mathrm{tr}(A)^2$ when $\mathrm{tr}(A)^2 - 8\mathrm{tr}(A) + 12 = 0$. This occurs when $\mathrm{tr}(A) = 2$ and $\mathrm{tr}(A) = 6$, corresponding to $c = 0$ and $c = 4$. Therefore during this time we are an unstable spiral. Before this, we pass through the $\mathrm{tr}(A)$-axis at $c = -1/2$, so we have the following classification: the origin is an unstable spiral if $0 < c < 4$, an unstable node if $-1/2 < c < 0$ or $c > 4$ and a saddle if $c < -1/2$.

Problem 4 *Nonhomogeneous Equations I*

Determine the solution of the initial value problem

$$\frac{d}{dx}\vec{y} = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \quad \vec{b}(x) = \begin{pmatrix} e^{2x} \\ 0 \end{pmatrix}, \; \vec{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

. . . .

Solution 4. The eigenvalues of A are -2 and 4, so using our matrix exponential tricks we calculate a fundamental matrix:

$$\begin{aligned}\Phi(x) = \exp(Ax) &= (A + 2I)\frac{1}{4+2}e^{4x} + (A - 4I)\frac{1}{-2-4}e^{-2x} \\ &= \begin{pmatrix} (1/2)e^{4x} + (1/2)e^{-2x} & (1/2)e^{4x} - (1/2)e^{-2x} \\ (1/2)e^{4x} - (1/2)e^{-2x} & (1/2)e^{4x} + (1/2)e^{-2x} \end{pmatrix}\end{aligned}$$

Furthermore since 2 is not an eigenvalue of the matrix, we may use the method of undetermined coefficients to find a solution. To do so, we propose a solution of the form $\vec{y}_p = \vec{v}e^{2x}$. Then we calculate

$$2\vec{v}e^{2x} = A\vec{v}e^{2x} + \begin{pmatrix} e^{2x} \\ 0 \end{pmatrix}.$$

Dividing out by e^{2x} and simplifying, we obtain:

$$(A - 2I)\vec{v} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Multiplying both sides by the inverse of $A - 2I$, we obtain $\vec{v} = \binom{-1/8}{-3/8}$. Therefore a particular solution is $\vec{y}_p = \binom{-1/8}{-3/8}e^{2x}$. The general solution is therefore

$$\vec{y} = \vec{y}_p + \vec{y}_h = \begin{pmatrix} -1/8 \\ -3/8 \end{pmatrix} e^{2x} + \Phi(x)\vec{c}.$$

Now plugging in our initial condition, we obtain (using the fact that $\Phi(0) = I$)

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/8 \\ -3/8 \end{pmatrix} + I\vec{c}.$$

Therefore $\vec{c} = \binom{9/8}{11/8}$, and the solution of the IVP is

$$\vec{y} = \vec{y}_p + \vec{y}_h = \begin{pmatrix} -1/8 \\ -3/8 \end{pmatrix} e^{2x} + \Phi(x)\begin{pmatrix} 9/8 \\ 11/8 \end{pmatrix}.$$

Problem 5 *Nonhomogeneous Equations II*

Determine the solution of the initial value problem

$$\frac{d}{dx}\vec{y} = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \quad \vec{b}(x) = \begin{pmatrix} e^{4x} \\ 0 \end{pmatrix}, \ \vec{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

. . . .

Solution 5. We determined the eigenvalues and found the fundamental matrix for this equation last time, so to start out we need to find a particular solution. The method of undetermined coefficients won't work here, since 4 is an eigenvalue of A. Therefore we should use something else, such as diagonalization. Since the eigenvalues of A are 4 and -2, our trick for finding eigenvectors for 2×2 matrices gives us an eigenvector $\binom{3}{3}$ for eigenvalue 4 and an eigenvector -33 for eigenvalue -2. Therefore we should take $P = \begin{pmatrix} 3 & -3 \\ 3 & 3 \end{pmatrix}$, $N = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$. Then $P^{-1} = \frac{1}{18}\begin{pmatrix} 3 & 3 \\ -3 & 3 \end{pmatrix}$, and substituting $\vec{y} = P\vec{z}$, our system reduces to

$$\frac{d}{dx}\vec{z} = N\vec{z} + P^{-1}\vec{b}$$

Furthermore, we calculate $P^{-1}\vec{b} = \frac{1}{18}\binom{3}{-3}e^{4x}$ and letting $\vec{z} = \binom{z_1}{z_2}$ the above reduces to the system of two ordinary equations

$$z_1' = 4z_1 + \frac{3}{18}e^{4x}, \qquad z_2' = -2z_2 - \frac{3}{18}e^{4x}.$$

Using the method of integrating factors, we find solutions:

$$z_1 = \frac{3}{18}xe^{4x}, \quad z_2 = -\frac{1}{36}e^{4x}.$$

Then

$$\vec{y}_p = P\vec{z} = P\begin{pmatrix} 3x/18 \\ -1/36 \end{pmatrix} e^{4x} = \begin{pmatrix} x/2 + 1/12 \\ x/2 - 1/12 \end{pmatrix} e^{4x}.$$

Then using the value of Φ from the last problem, the general solution is seen to be

$$\vec{y} = \vec{y}_p + \vec{y}_h = \begin{pmatrix} x/2 + 1/12 \\ x/2 - 1/12 \end{pmatrix} e^{4x} + \Phi(x)\vec{c}.$$

Plugging in our initial conditions, we obtain

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/12 \\ -1/12 \end{pmatrix} + \vec{c},$$

and therefore $\vec{c} = \binom{-1/12}{13/12}$. The solution of the initial value problem is therefore

$$\vec{y} = \vec{y}_p + \vec{y}_h = \begin{pmatrix} x/2 + 1/12 \\ x/2 - 1/12 \end{pmatrix} e^{4x} + \Phi(x)\begin{pmatrix} -1/12 \\ 13/12 \end{pmatrix}.$$

Problem 6 *Matrices with One Eigenvalue*

Let A be a 2×2 matrix, and suppose that A has exactly one eigenvalue λ with algebraic multiplicity 2. In this problem, we will show that

$$\exp(Ax) = Ie^{\lambda x} + (A - \lambda I)xe^{\lambda x} \tag{1}$$

(a) Define the matrix $N = (A - \lambda I)$. Show that $N^2 = 0$.

(b) Show that since $N^2 = 0$, we have $\exp(Nx) = I + Nx$

(c) Complete the proof of Equation (1) by using Proposition (1) below.

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Solution 6.

(a) Since A has an eigenvalue λ with algebraic multiplicity 2, there are only two possible Jordan normal forms for A, namely $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ or $J = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Furthermore, there exists some matrix P satisfying $P^{-1}AP = J$. It follows that

$$P^{-1}NP = P^{-1}(A - \lambda I)P = P^{-1}AP - P^{-1}(\lambda I)P = P^{-1}AP - \lambda P^{-1}IP = J - \lambda I$$

Note that $J - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $J - \lambda I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. In either case $(J - \lambda I)^2 = 0$, and therefore

$$(P^{-1}NP)^2 = (J - \lambda I)^2 = 0.$$

However, $(P^{-1}NP)^2 = P^{-1}N^2P$, so this shows that $P^{-1}N^2P = 0$. Multiplying by P on the left and P^{-1} on the right, thish shows that $N^2 = 0$.

(b) By definition,

$$\exp(Nx) = I + Nx + \frac{1}{2}N^2x^2 + \frac{1}{3!}N^3x^3 + \cdots = I + Nx + \frac{1}{2}0x^2 + \frac{1}{3!}0x^3 + \cdots = I + NX.$$

(c) Since $I\lambda$ and N commute and $A = N + \lambda I$, the Proposition below tells us that

$$\exp(Ax) = \exp(Nx + \lambda Ix) = \exp(Nx)\exp(\lambda Ix) = (I + Nx)\exp(\lambda x)I.$$

Then replacing N with its value $A - \lambda I$ gives us the identity we wanted.

Proposition 1. Suppose that B, C are two $n \times n$ matrices which commute, ie. $AB = BA$. Then

$$\exp(Ax + Bx) = \exp(Ax)\exp(Bx).$$