MATH 309: Homework #4

Due on: May 6, 2016

Problem 1 *Fourier Series*

For each of the following functions, sketch a graph of the function and find the Fourier series

(a) $f(x) = \sin^3(x) + \cos^2(2x+3)$

(b) $f(x) = -x, \ -L \leq x < L$ with $f(x+2L) = f(x)$ for all x

(c) $f(x) = \begin{cases} x+1, & -\pi \leq x < 0 \\ 1-x, & 0 \leq x < \pi \end{cases}$ with $f(x+2\pi) = f(x)$ for all x

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Solution 1. We will omit the sketches, as we assume that students are able to figure that part out.

(a) The idea of this first problem is to use a little bit of trigonometry to write $f(x)$ as a finite sum of sines and cosines. This is easier here than trying to apply the Euler-Fourier formula directly. The triggy-tricks that we will use are the following:

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta).$$

$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi).$$

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi).$$

$$\sin(\theta)\cos(\phi) = \frac{1}{2}\left(\sin(\theta+\phi) + \sin(\theta - \phi)\right).$$

For starters:

$$\begin{aligned}
\cos^2(2x+3) &= (1/2) + (1/2)\cos(4x+6) \\
&= (1/2) + (1/2)[\cos(4x)\cos(6) - \sin(4x)\sin(6)] \\
&= (1/2) + \frac{1}{2}\cos(6)\cos(4x) - \frac{1}{2}\sin(6)\sin(4x).
\end{aligned}$$

Moreover:

$$\begin{aligned}
\sin^3(x) &= (1-\cos^2(x))\sin(x) \\
&= ((1/2)-(1/2)\cos(2x))\sin(x) \\
&= (1/2)\sin(x)-(1/2)\cos(2x)\sin(x) \\
&= (1/2)\sin(x)-(1/4)(\sin(3x)-\sin(x)) \\
&= \frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x).
\end{aligned}$$

Therefore we have that $a_0 = 1, b_1 = 3/4, b_3 = 1/4, a_4 = \cos(6)/2, b_4 = \sin(6)/2$ and a_i, b_j are zero otherwise. In other words

$$f(x) = (1/2) + \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x) + \frac{1}{2}\cos(6)\cos(4x) - \frac{1}{2}\sin(6)\sin(4x).$$

(b) The second function is a "sawtooth" wave. Note that $f(x)$ is odd, forcing $a_n = 0$ for all n. Therefore we need only worry about the b_n's. For these, we apply the Euler-Fourier formula:

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{L}\int_{-L}^{L} -x\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)|_{-L}^{L} - \frac{1}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2L}{n\pi}\cos(n\pi) - \frac{1}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2L}{n\pi}\cos(n\pi) - \frac{1}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2L}{n\pi}\cos(n\pi) - \frac{L}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)|_{-L}^{L} = \frac{2L}{n\pi}\cos(n\pi).
\end{aligned}$$

Then since $\cos(n\pi) = (-1)^n$, we see that

$$f(x) = \sum_{n=1}^{\infty}\frac{2L(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{L}\right).$$

(c) The third function is a "triangular wave". Note that $f(x)$ is even, forcing $b_n = 0$ for all n. Therefore we need only worry about the a_n's. For these, we apply the

Euler-Fourier formula:

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi}(1-x)\cos(nx)dx \\
&= \frac{2}{n\pi}(1-x)\sin(nx)|_0^\pi + \frac{2}{n\pi}\int_0^\pi \sin(nx)dx \\
&= \frac{2}{n\pi}\int_0^\pi \sin(nx)dx = -\frac{2}{n^2\pi}\cos(nx)|_0^\pi \\
&= -\frac{2}{n^2\pi}(\cos(n\pi)-1) = -\frac{2}{n^2\pi}((-1)^n-1).
\end{aligned}
$$

In particular, $a_n = 0$ when n is even. Note that in the above calculation, we used the fact that $n \neq 0$. We need to do the case $n = 0$ separately:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cos(0\pi x/L)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = -(\pi-2).$$

Putting this all together we have

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n-1}\cos((2n-1)x) \\
&= \frac{2-\pi}{2} + \sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)^2}\cos((2n-1)x)
\end{aligned}
$$

Problem 2 *Parseval's Identity*

Let $f(x)$ be a periodic function with fundamental period $2L$, and suppose that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

Using the fact that

$$\left\{\frac{1}{2}, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) : n = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots\right\}$$

is a mutually orthogonal set of functions, prove Parseval's identity:

$$\frac{1}{L}\int_{-L}^{L} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

. . . .

Solution 2. This problem is easier to understand if we use the inner product notation

$$\langle g(x), h(x)\rangle = \int_{-L}^{L} g(x), h(x) dx.$$

Then using the linearity of the inner product, we have that

$$\begin{aligned}\int_{-L}^{L} f(x)^2 dx &= \langle f, f\rangle \\ &= \left\langle f(x), \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]\right\rangle \\ &= a_0 \left\langle f(x), \frac{1}{2}\right\rangle + \sum_{n=1}^{\infty} a_n \left\langle f(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle + \sum_{n=1}^{\infty} b_n \left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle.\end{aligned}$$

For fixed n, we calculate using orthogonality:

$$\begin{aligned}\left\langle f(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle &= \left\langle \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right], \cos\left(\frac{n\pi x}{L}\right)\right\rangle \\ &= \left\langle \frac{a_0}{2}, \cos\left(\frac{n\pi x}{L}\right)\right\rangle + \sum_{m=1}^{\infty} a_m \left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\rangle + \sum_{m=1}^{\infty} b_m \left\langle \sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\rangle \\ &= a_m \left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\rangle = a_m L.\end{aligned}$$

A similar calculation also shows

$$\left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle = b_n L.$$

and that

$$\left\langle f(x), \frac{1}{2}\right\rangle = \frac{1}{2} a_0 L.$$

Therefore we see that

$$\begin{aligned}&\int_{-L}^{L} f(x)^2 dx \\ &\quad = a_0 \left\langle f(x), \frac{1}{2}\right\rangle + \sum_{n=1}^{\infty} a_n \left\langle f(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle + \sum_{n=1}^{\infty} b_n \left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle \\ &\quad = a_0(a_0 L/2) + \sum_{n=1}^{\infty} a_n(a_n L) + \sum_{n=1}^{\infty} b_n(b_n L) = \frac{1}{2} a_0^2 L + \sum_{n=1}^{\infty} \left(a_n^2 L + b_n^2 L\right)\end{aligned}$$

Dividing now by L gives us Parseval's identity.

Problem 3 *Parseval's Identity Application*

Use Parseval's identity and the Fourier series for the square wave function

$$f(x) = \begin{cases} 0, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1 \end{cases}, \text{ with } f(x+2) = f(x) \text{ for all } x$$

to obtain the value of the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

. . . .

Solution 3. We first calculate the Fourier series for the square wave function above using the Euler-Fourier formula. We calculate

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x) = \int_{0}^{1} \cos(n\pi x) = \frac{1}{n\pi}\sin(n\pi x)|_0^1 = 0.$$

The above calculation does not work when $n = 0$ however (since we divided by n). We have to do this separately:

$$a_0 = \int_{-1}^{1} f(x)dx = 1.$$

We also calculate the b_n's:

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x) = \int_{0}^{1} \sin(n\pi x) = -\frac{1}{n\pi}\cos(n\pi x)|_0^1 = -\frac{1}{n\pi}((-1)^n - 1).$$

This last expression is 0 if n is even and 1 if n is odd. Therefore we have that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(n\pi x) \qquad = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_{2n-1}\sin(n\pi x)$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi}\sin(n\pi x).$$

Then since

$$\frac{1}{1}\int_{-1}^{1} f(x)^2 dx = 1,$$

Parseval's identity tells us that

$$1 = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2}.$$

Simplifying this a bit, it says

$$\frac{1}{2} = \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2},$$

and therefore
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$
We should take a minute at this point to pause appreciatively, since we have shown that the sum of the reciprocals of the positive odd integers is related to π!