MATH 309: Homework #5

Due on: May 18, 2016

Problem 1 *Boundary Value Problems*

For each of the following boundary value problems, find all solutions to the boundary value problem or show that no solution exists.

(a) $y'' + y = 0$, $y(0) = 0$, $y'(\pi) = 1$

(b) $y'' + y = 0$, $y(0) = 0$, $y(L) = 0$

(c) $y'' + y = x$, $y(0) = 0$, $y(\pi) = 0$

. . . .

Solution 1. In each case, the general solution is

$$y(x) = A\cos(x) + B\sin(x),$$

so the question is whether or not we can find constants A, B satisfying the boundary conditions.

(a) The condition $y(0) = 0$ implies that $A = 0$. Therefore $y(x) = B\sin(x)$. The condition $y'(\pi) = 0$ implies that $B = 0$, and therefore the only solution is the trivial solution $y = 0$.

(b) The condition $y(0) = 0$ implies that $A = 0$. Therefore $y(x) = B\sin(x)$. The condition $y(L) = 0$ implies that $B\sin(L) = 0$, and therefore either $B = 0$, giving us the trivial solution, or else $L = n\pi$ for some integer n, in which case B can be anything! Thus we have two cases: if L is not an integer multiple of π, then the only solution is the trivial solution $y = 0$. If $L = n\pi$ for some integer n, then the family of all solutions is $y = B\sin(x)$.

(c) The condition $y(0) = 0$ implies that $A = 0$. Therefore $y(x) = B\sin(x)$, therefore the condition $y(\pi) = 0$ is automatically satisfied, leaving implies that $B = 0$, and therefore the only solution is the trivial solution $y = 0$.

Problem 2 *Dirichlet Eigenvalue Problem*

Determine for which values of λ the boundary value problem

$$y'' + \lambda y = 0,\ y(0) = 0,\ y(L) = 0,$$

has a solution and describe the solutions.

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Solution 2. It's important to note that the values of λ which work will be dependent on the value of L – this relationship between λ and L becomes important in the method of separation of variables later on. Let's first think about the general solution to $y'' + \lambda y$. The characteristic polynomial of this equation is $x^2 + \lambda$, which has roots $\pm\sqrt{-\lambda}$. The general solution therefore takes three distinct forms, depending on whether λ is positive, negative, or zero.

Case A ($\lambda < 0$):
In this case, $\sqrt{-\lambda}$ is real, so the general solution is

$$y = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

Then since $y(0) = 0$, we have $A + B = 0$. Furthermore, since $y(L) = 0$ we have $Ae^{\sqrt{-\lambda}L} + Be^{-\sqrt{-\lambda}L} = 0$. Thus we have a homogeneous system of two equations and two unknowns. In matrix form, this is

$$\begin{pmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}L} & e^{-\sqrt{-\lambda}L} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the above matrix is $e^{-\sqrt{-\lambda}L} - e^{\sqrt{-\lambda}L}$, which is nonzero. Therefore the matrix is nonsingular, and the homogeneous system of equations has exactly one solution: the trivial solution. Therefore $A = B = 0$, making $y = 0$ the only solution to the boundary value problem.

Case B ($\lambda = 0$):
In this case, $\sqrt{-\lambda}$ is 0, so the general solution is

$$y = A + Bx.$$

Then since $y(0) = 0$, we have $A = 0$. Furthermore, since $y(L) = 0$ we have $A + BL = 0$. Since $A = 0$, this also says that $B = 0$, and therefore the only solution is the trivial solution $y = 0$.

Case C ($\lambda > 0$):
In this case, $\sqrt{-\lambda} = i\sqrt{\lambda}$ is imaginary, so the general solution is

$$y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Then since $y(0) = 0$, we have $A = 0$, making $y = B\sin(\sqrt{\lambda}x)$. Then since $y(L) = 0$, we have that $B = 0$ or $\sin(\sqrt{\lambda}L) = 0$. In the former case, $y = 0$. In the latter

case, $\sqrt{\lambda}L = n\pi$ for some integer n and therefore $\lambda = n^2\pi^2/L^2$. In this case $y = B\sin(\sqrt{\lambda}x) = B\sin(n\pi x/L)$ is a solution for any value of B.
SUMMARY:
The boundary value problem has at least one solution for every value of λ: the trivial solution. The boundary value problem has more than the trivial solution exactly when $\lambda = n^2\pi^2/L^2$ for some nonzero integer n, in which case anything of the form $B\sin(n\pi x/L)$ is a solution.

Problem 3 *Neumann Eigenvalue Problem*

Determine for which values of λ the boundary value problem

$$y'' + \lambda y = 0,\ y'(0) = 0,\ y'(L) = 0,$$

has a solution and describe the solutions.

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Solution 3. It's important to note that the values of λ which work will be dependent on the value of L – this relationship between λ and L becomes important in the method of separation of variables later on. Let's first think about the general solution to $y'' + \lambda y$. The characteristic polynomial of this equation is $x^2 + \lambda$, which has roots $\pm\sqrt{-\lambda}$. The general solution therefore takes three distinct forms, depending on whether λ is positive, negative, or zero.
Case A ($\lambda < 0$):
In this case, $\sqrt{-\lambda}$ is real, so the general solution is

$$y = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

We note that

$$y' = \sqrt{-\lambda}(Ae^{\sqrt{-\lambda}x} - Be^{-\sqrt{-\lambda}x}).$$

Then since $y'(0) = 0$, we have $A - B = 0$. Furthermore, since $y'(L) = 0$ we have $Ae^{\sqrt{-\lambda}L} - Be^{-\sqrt{-\lambda}L} = 0$. Thus we have a homogeneous system of two equations and two unknowns. In matrix form, this is

$$\begin{pmatrix} 1 & -1 \\ e^{\sqrt{-\lambda}L} & -e^{-\sqrt{-\lambda}L} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the above matrix is $e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}$, which is nonzero. Therefore the matrix is nonsingular, and the homogeneous system of equations has exactly one solution: the trivial solution. Therefore $A = B = 0$, making $y = 0$ the only solution to the boundary value problem.
Case B ($\lambda = 0$):
In this case, $\sqrt{-\lambda}$ is 0, so the general solution is

$$y = A + Bx.$$

We note that

$$y' = B$$

Then since $y'(0) = 0$, we have $B = 0$. Furthermore, since $y'(L) = 0$ we have $B = 0$, again. Thus $y = A$ is a solution for any value of A. **Case C** ($\lambda > 0$):
In this case, $\sqrt{-\lambda} = i\sqrt{\lambda}$ is imaginary, so the general solution is

$$y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

We note that

$$y' = \sqrt{\lambda x}(B\cos(\sqrt{\lambda}x) - A\sin(\sqrt{\lambda}x)).$$

Then since $y'(0) = 0$, we have $B = 0$, making $y = A\cos(\sqrt{\lambda}x)$. Then since $y'(L) = 0$, we have that $A = 0$ or $\sin(\sqrt{\lambda}L) = 0$. In the former case, $y = 0$. In the latter case, $\sqrt{\lambda}L = n\pi$ for some integer n and therefore $\lambda = n^2\pi^2/L^2$. In this case $y = A\cos(\sqrt{\lambda}x) = A\cos(n\pi x/L)$ is a solution for any value of B.
SUMMARY:
The boundary value problem has at least one solution for every value of λ: the trivial solution. The boundary value problem has more than the trivial solution exactly when $\lambda = 0$ or $\lambda = n^2\pi^2/L^2$ for some nonzero integer n. If $\lambda = 0$, then anything of the form $y = A$ is a solution. If $\lambda = n^2\pi^2/L^2$, then anything of the form $y = A\cos(n\pi x/L)$ is a solution.

Problem 4 *Even and Odd Functions*

Prove that any function $f(x)$ may be expressed as a sum of two functions $f(x) = g(x) + h(x)$ with $g(x)$ even and $h(x)$ odd. [Hint: consider $f(x) + f(-x)$].

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Solution 4. In order to prove the statement we want, we need to show that for any function $f(x)$, there exists an even function $g(x)$ and an odd function $h(x)$ with $f(x) = g(x) + h(x)$. In particular, we need to come up with equations for $g(x)$ and $h(x)$ in terms of $f(x)$. How can we do this? One way is to assume that $g(x)$ and $h(x)$ are known to exist, and then fiddle around with $f(x)$ to figure out the equations. In particular if $g(x)$ is even and $h(x)$ is odd and $f(x) = g(x) + h(x)$ then

$$f(-x) = g(-x) + h(-x) = g(x) - h(x).$$

It follows that

$$f(x) + f(-x) = g(x) + h(x) + (g(x) - h(x)) = 2g(x),$$

and therefore we should take $g(x) = (f(x) + f(-x))/2$. Similarly, we have that

$$f(x) - f(-x) = g(x) + h(x) - (g(x) - h(x)) = 2h(x),$$

and therfore we should take $h(x) = (f(x) - f(-x))/2$. Great!

What we did above is just a bunch of scratch work. Here's our actual proof: Suppose that $f(x)$ is a function. Define $g(x) = (f(x)+f(-x))/2$ and $h(x) = (f(x)-f(-x))/2$. Then since

$$g(-x) = (f(-x) + f(--x))/2 = (f(-x) + f(x))/2 = (f(x) + f(-x))/2 = g(x)$$

we have that $g(x)$ is even. Similarly

$$h(-x) = (f(-x) - f(--x))/2 = (f(-x) - f(x))/2 = -(f(x) - f(-x))/2 = -h(x)$$

and therefore $h(x)$ is odd. Furthermore

$$g(x) + h(x) = (f(x) + f(-x))/2 + (f(x) - f(-x))/2 = f(x).$$

Therefore $f(x) = g(x) + h(x)$ is a sum of an even function and an odd function. This completes our proof.

Problem 5 *Even and Odd Functions*

Prove that the derivative of an even function is odd and that the derivative of an odd function is even.

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Solution 5. There are many great ways to prove this fact. We will use one of the simplest methods: the chain rule. Let $g(x) = f(-x)$. Then by the chain rule

$$g'(x) = -f'(-x).$$

Now let's suppose $f(x)$ is an even function. Then in this case $g(x) = f(x)$, making $g'(x) = f'(x)$, so that the above expression reads $f'(x) = -f'(-x)$. Since x was arbitrary, this shows that $f'(x)$ is odd when $f(x)$ is even. Alternatively, let's suppose that $f(x)$ is an odd function. Then $g(x) = -f(x)$, making $g'(x) = -f'(x)$, so that the expression we derived from the chain rule reads $-f'(x) = -f'(-x)$, and hence $f'(x) = f'(-x)$. Since x was arbitrary, this shows that $f'(x)$ is even when $f(x)$ is odd. This completes our proof.

Problem 6 *Sine Series*

Consider the function

$$f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \\ 2, & 2\pi < x < 3\pi \end{cases}$$

(a) Scketch a graph of $f(x)$

(b) By reflecting f appropriately, express f as a sine series.

(c) Plot three different partial sums of the sine series, clearly indicating the partial sums being plotted.

(d) Sketch a graph of the function to which the sine series converges for three periods.

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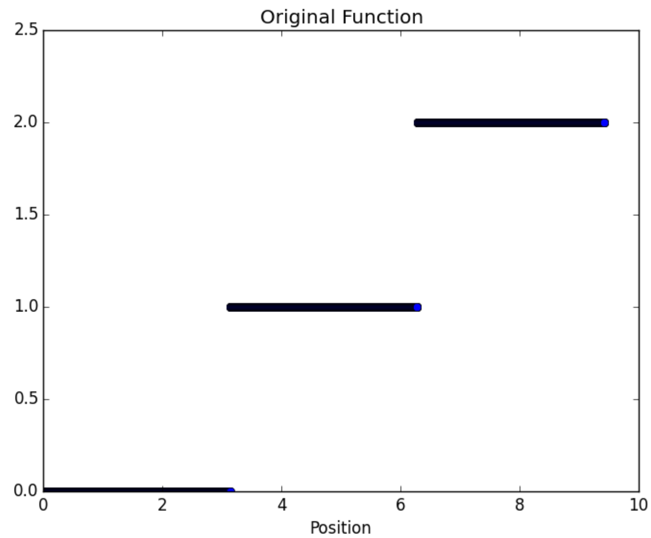
Solution 6.



(a)

(b) To express $f(x)$ as a sine series, we create a new function $g(x)$ which is odd and periodic by reflecting $f(x)$ oddly accross the y-axis, and then defining $g(x+6\pi) = g(x)$ for all x. Since $g(x)$ is periodic, it has a Fourier series, and since $g(x)$ is odd, all of the cosine terms will be gone, leaving just the sine terms. We can calculate the associated coefficients by using the Euler-Fourier formula:

$$b_n = \frac{1}{3\pi}\int_{-3\pi}^{3\pi} g(x)\sin(n\pi x/(3\pi))dx.$$

Now since $g(x)$ is odd, the integrand is even, so we can simply integrate from 0 to 3π and multiply by 2 to get the value of b_n. Moreover, from 0 to 3π the function $g(x)$ agrees with $f(x)$, and therefore

$$b_n = \frac{2}{3\pi}\int_0^{3\pi} f(x)\sin(nx/3)dx.$$

Now in order to do this intergral, we need to break it up into the three separate intervals where $f(x)$ is individually defined:

$$b_n = \frac{2}{3\pi}\left(\int_0^{\pi} 0\sin(nx/3) + \int_{\pi}^{2\pi} 1\sin(nx/3)dx + \int_{2\pi}^{3\pi} 3\sin(nx/3)dx\right).$$

The integrals themselves are pretty easy. Evaluating them, we obtain:

$$b_n = \frac{2}{3\pi}\left(0 + \frac{-3}{n}(\cos(2n\pi/3) - \cos(n\pi/3))dx + \frac{-9}{n}(\cos(3n\pi/3) - \cos(2n\pi/3))\right).$$

Now we want to use the fact that

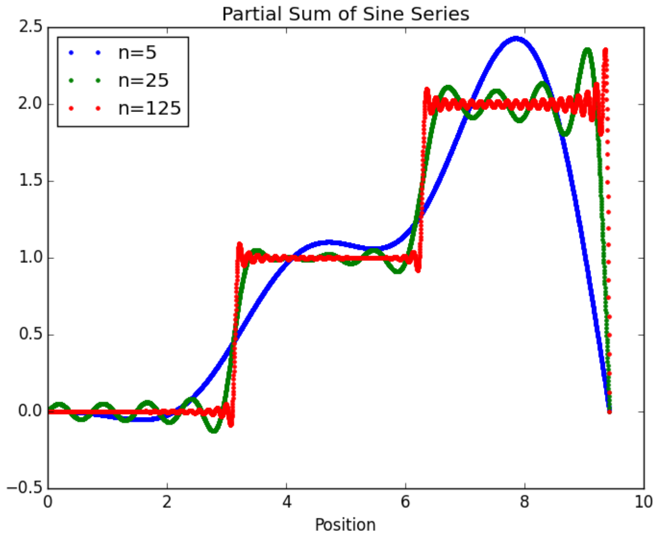
$$\cos(m\pi/3) = \begin{cases} 1/2, & m = \pm 1 + 6k \\ -1/2, & m = \pm 2 + 6k \\ 1, & m = 0 + 6k \\ -1, & m = 3 + 6k \end{cases}$$

Using this, the expression for b_n reduces to

$$b_n = \begin{cases} 5/(n\pi), & n = \pm 1 + 6k \\ -9/(n\pi), & n = \pm 2 + 6k \\ 0, & n = 0 + 6k \\ 8/(n\pi), & n = 3 + 6k \end{cases}$$

Using these values of b_n, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx/3).$$



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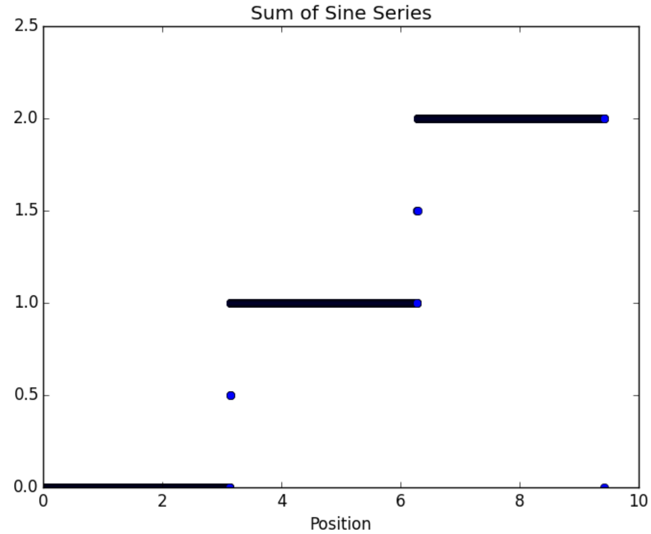

(d)

Problem 7 *Cosine Series*

Consider the function

$$f(x) = \begin{cases} x, 0 < x < \pi \\ 0, \pi < x < 2\pi \end{cases}$$

(a) Scketch a graph of $f(x)$

(b) By reflecting f appropriately, express f as a cosine series.

(c) Plot three different partial sums of the cosine series, clearly indicating the partial sums being plotted.

(d) Sketch a graph of the function to which the cosine series converges for three periods.

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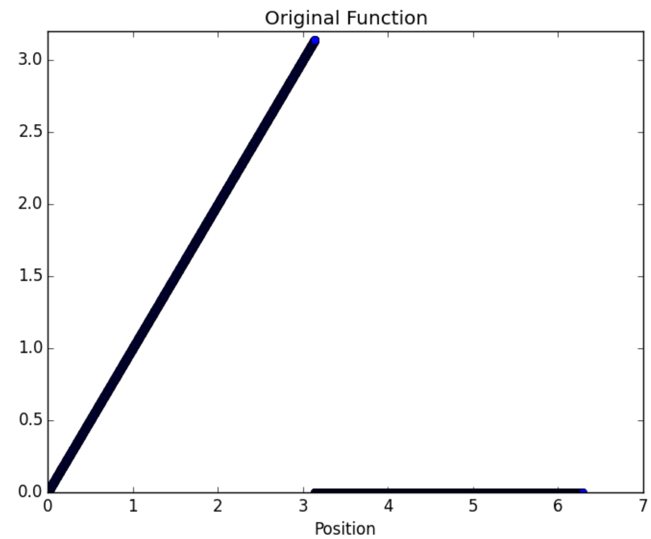
Solution 7.

(a)



(b) To express $f(x)$ as a cosine series, we create a new function $g(x)$ which is even and periodic by reflecting $f(x)$ evenly accross the y-axis, and then defining $g(x+4\pi) = g(x)$ for all x. Since $g(x)$ is periodic, it has a Fourier series, and since $g(x)$ is even, all of the sine terms will be gone, leaving just the cosine terms. We can calculate the associated coefficients by using the Euler-Fourier formula:

$$a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} g(x)\cos(n\pi x/(2\pi))dx.$$

Now since $g(x)$ is odd, the integrand is even, so we can simply integrate from 0 to 2π and multiply by 2 to get the value of a_n. Moreover, from 0 to 2π the function $g(x)$ agrees with $f(x)$, and therefore

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx/2)dx.$$

Now in order to do this intergral, we need to break it up into the two separate intervals where $f(x)$ is individually defined:

$$a_n = \frac{1}{\pi}\left(\int_0^{\pi} x\cos(nx/2) + \int_{\pi}^{2\pi} 0\cos(nx/2)dx\right).$$

To evaluate this integral, we use integration by parts, obtaining:

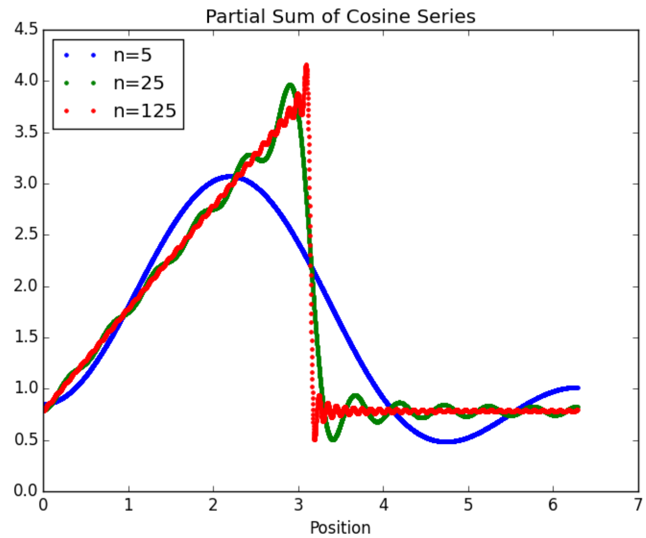
$$a_n = \frac{-2}{n}\cos(n\pi/2) = \begin{cases} ((-1)^{n/2}-1)4/(n^2\pi) & n \text{ even} \\ (-1)^{(n+1)/2}2/n - 4/(n^2\pi) & n \text{ odd} \end{cases}$$

This expression does not work however for $n = 0$ since in the calculation we divided by n. We must do this separately:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} xdx = \frac{1}{2}\pi.$$

Using these values of a_n, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/3).$$



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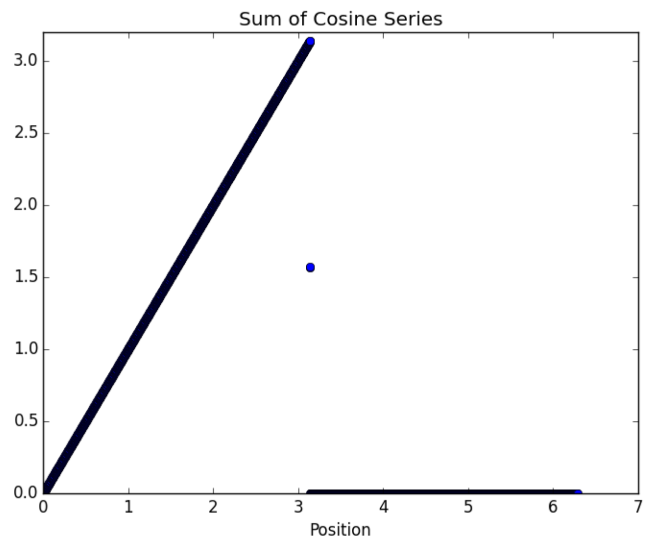

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