MATH 309: Homework #7

Due on: June 1, 2016

Problem 1 *The Heat Equation in Two Dimensions*

We consider the two dimensional heat equation

$$u_t - \alpha^2(u_{xx} + u_{yy}) = 0.$$

(a) Assume that u is of the form $u(x, y, t) = F(x)G(y)T(t)$, and show that the heat equation reduces to the system of three ordinary differential equations

$$\begin{cases} T'(t) + \lambda T = 0 \\ F''(x) + \frac{\lambda - \mu}{\alpha^2} F(x) = 0 \\ G''(y) + \frac{\mu}{\alpha^2} G(y) = 0 \end{cases}$$

for some constants λ and μ.

(b) Assume that $u(x, y, t) = F(x)G(y)T(t)$ satisfies the heat equation above in the rectangular region $[0, L] \times [0, M]$ and also satisfies the Dirichlet boundary conditions

$$u(0, y, t) = 0, u(L, y, t) = 0, u(x, 0, t) = 0, u(x, M, t) = 0.$$

Find all possible functions $u(x, y, t)$ satisfying the above conditions. [Hint: they should be indexed by pairs of positive integers (m, n)]

(c) Use (b) to find a solution to the two dimensional heat equation with Dirichlet boundary conditions

$$u_t - (u_{xx} + u_{yy}) = 0,$$

$$u(0, y, t) = 0, u(1, y, t) = 0, u(x, 0, t) = 0, u(x, 1, t) = 0,$$

with the initial condition that

$$u(x, y, 0) = \sin(3\pi x)\sin(2\pi y) + \sin(2\pi x)\sin(4\pi y).$$

Create a surface plots of your solution for several values of t.

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Problem 2 *The Heat Equation in Polar Coordinates*

We consider the two dimensional heat equation

$$u_t - \alpha^2(u_{xx} + u_{yy}) = 0.$$

(a) Show that using polar coordinates, (r, θ), the heat equation becomes

$$u_t - \alpha^2 \left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \right) = 0.$$

(b) Assume that u is of the form $u(r, \theta, t) = R(r)S(\theta)T(t)$, and show that the heat equation reduces to the system of three ordinary differential equations

$$\left\{ \begin{array}{c} T'(t) + \lambda T = 0 \\ r^2 R''(r) + rR'(r) + \frac{1}{\alpha^2}(r^2\lambda - \mu)R = 0 \\ S''(\theta) + \frac{\mu}{\alpha^2}S(\theta) = 0 \end{array} \right.$$

for some constants λ and μ.

(c) Explain why $\mu = n^2\alpha^2$ for some integer n. [Hint: remember that θ is the angle counter-clockwise from the x-axis].

(d) Find the general solution to the above system of equations in the case that $\lambda = 0$ and $\mu = \alpha^2$. [Hint: to solve for $R(r)$, propose a solution of the form $R(r) = r^b$]

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Problem 3 *The Wave Equation I*

Consider an elastic string of length $L = 10$ whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x, 0) = f(x)$, and the material properties of the string make $u(x, t)$ satisfy the wave equation $u_{tt} - c^2 u_{xx}$ with $c = 1$. For each of the values of $f(x)$ below, determine

(i) Determine the solution $u(x, t)$ in terms of an infinite linear combination of the fundamental set of solutions $u_n(x, t) = \sin(n\pi x/L)\cos(cn\pi t/L)$

(ii) Plot $u(x, t)$ vs. x for $t = 0, 4, 8, 12, 16$

(iii) Describe the motion of the string in a few sentences.

(a)

$$f(x) = \left\{ \begin{array}{ll} 2x/L, & 0 \le x \le L/2 \\ 2(L - x)/L, & L/2 < x \le L \end{array} \right.$$

(b)

$$f(x) = 8x(L - x)^2/L^3.$$

(c)

$$f(x) = \left\{ \begin{array}{ll} 1, & |x - L/2| < 1 \\ 0, & |x - L/2| \ge 1 \end{array} \right.$$

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Problem 4 *The Wave Equation II*

Consider an elastic string of length $L = 10$ whose ends are held fixed. The string is set in motion from its equilibrium position with initial velocity given by $u_t(x,0) = g(x)$, and the material properties of the string make $u(x,t)$ satisfy the wave equation $u_{tt} - c^2 u_{xx}$ with $c = 1$. For each of the values of $g(x)$ below, determine

(i) Determine the solution $u(x,t)$ for $0 \le x \le L$ and $t > 0$ in terms of an infinite linear combination of the fundamental set of solutions $u_n(x,t) = \sin(n\pi x/L)\sin(cn\pi t/L)$

(ii) Plot $u(x,t)$ vs. x for $t = 0, 4, 8, 12, 16$

(iii) Describe the motion of the string in a few sentences.

(a)

$$g(x) = \begin{cases} 2x/L, & 0 \le x \le L/2 \\ 2(L-x)/L, & L/2 < x \le L \end{cases}$$

(b)

$$g(x) = 8x(L-x)^2/L^3.$$

(c)

$$g(x) = \begin{cases} 1, & |x - L/2| < 1 \\ 0, & |x - L/2| \ge 1 \end{cases}$$

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Problem 5 Some Physics Flavor

A steel wire 5 ft in length is stretched by a tensile force of 50 lb. The wire has a weight per unit lenght of 0.026 lb/ft.

(a) Find the velocity of propagation of transverse waves in the wire.

(b) Find the natrual frequencies of vibration.

(c) If the tension in the wire is increased, how are the natural frequencies changed? Are the natural modes also changed?

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Problem 6 D'Alembert's Method

Use D'Alembert's Method to find a solution to the wave equation

$$u_{tt} - u_{xx} = 0, \quad 0 \le x \le 1, \ t > 0$$

satisfying $u(0) = 0$ and $u(1) = 0$, with the property that $u(x,0) = \sin^3(\pi x)$ and $u_t(x,0) = 0$. Use this solution to create a surface plot of $u(x,t)$ for $0 \le x \le 1$ and $0 \le t \le 4$.

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Problem 7 Wave Equation with von Neummann Boundary Conditions

Use separation of variables to find a solution to the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

with the homogeneous von Neumman boundary conditions

$$u_x(0,t) = 0, \quad u_x(L,t) = 0,$$

and satisfying the initial condition

$$u(x,0) = \cos(n\pi x/L), u_t(x,0) = 0,$$

where here n is a nonnegative integer.

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