MATH 309: Homework #7

Due on: June 1, 2016

Problem 1 *The Heat Equation in Two Dimensions*

We consider the two dimensional heat equation

$$u_t - \alpha^2(u_{xx} + u_{yy}) = 0.$$

(a) Assume that u is of the form $u(x, y, t) = F(x)G(y)T(t)$, and show that the heat equation reduces to the system of three ordinary differential equations

$$\begin{cases} T'(t) + \lambda T = 0 \\ F''(x) + \frac{\lambda - \mu}{\alpha^2} F(x) = 0 \\ G''(y) + \frac{\mu}{\alpha^2} G(y) = 0 \end{cases}$$

for some constants λ and μ.

(b) Assume that $u(x, y, t) = F(x)G(y)T(t)$ satisfies the heat equation above in the rectangular region $[0, L] \times [0, M]$ and also satisfies the Dirichlet boundary conditions

$$u(0, y, t) = 0, u(L, y, t) = 0, u(x, 0, t) = 0, u(x, M, t) = 0.$$

Find all possible functions $u(x, y, t)$ satisfying the above conditions. [Hint: they should be indexed by pairs of positive integers (m, n)]

(c) Use (b) to find a solution to the two dimensional heat equation with Dirichlet boundary conditions

$$u_t - (u_{xx} + u_{yy}) = 0,$$

$$u(0, y, t) = 0, u(1, y, t) = 0, u(x, 0, t) = 0, u(x, 1, t) = 0,$$

with the initial condition that

$$u(x, y, 0) = \sin(3\pi x)\sin(2\pi y) + \sin(2\pi x)\sin(4\pi y).$$

Create a surface plots of your solution for several values of t.

....

Solution 1.

(a) Pluggin $u(x, y, t) = F(x)G(y)T(t)$ into the two dimensional heat equation, we obtain
$$F(x)G(y)T'(t) = \alpha^2(F''(x)G(y)T(t) + F(x)G''(y)T(t)).$$
Then dividing on both sides by $F(x)G(y)T(t)$, we obtain:
$$T'(t)/T(t) = \alpha^2(F''(x)/F(x) + G''(y)/G(y)).$$
The expression on the left hand side is a function of t only, while the expression on the right hand side is independent of t. Therefore both must be equal to a constant $-\lambda$:
$$T'(t)/T(t) = \alpha^2(F''(x)/F(x) + G''(y)/G(y)) = -\lambda.$$
Simplifying this, we get
$$T'(t)/T(t) = -\lambda$$
$$\alpha^2 F''(x)/F(x) + \alpha^2 G''(y)/G(y) = -\lambda.$$
Therefore
$$\alpha^2 F''(x)/F(x) = -\alpha^2 G''(y)/G(y) - \lambda.$$
Again the expression on the left hand side is a function of x only, and on the right we have a function of y only, and therefore both are equal to a constant μ:
$$\alpha^2 F''(x)/F(x) + \lambda = -\alpha^2 G''(y)/G(y) = \mu.$$
It follows that
$$\alpha^2 F''(x)/F(x) = \mu - \lambda$$
$$-\alpha^2 G''(y)/G(y) = -\mu.$$
Simplyfying things further we obtain the system of three ordinary differential equations listed in (a) above.

(b) The Dirichlet boundary conditions result in boundary conditions on our various ODE's. In particular
$$F''(x) + \frac{\lambda - \mu}{\alpha^2}F(x) = 0, \ F(0) = 0, \ F(L) = 0$$
and also
$$G''(y) + \frac{\mu}{\alpha^2}G(y) = 0, \ G(0) = 0, \ G(M) = 0.$$
Then from our experience with boundary value problems, to get a nontrivial solution this says that $\mu = n^2\pi^2\alpha^2/M^2$ and that $\lambda - \mu = m^2\pi^2\alpha^2/L^2$ for some integers m and n, and in this case the solutions we obtain are
$$F(x) = A\sin(m\pi x/L), \quad G(x) = B\sin(n\pi y/M)$$

for some constants A and B. Then $\lambda = m^2\pi^2\alpha^2/L^2 + n^2\pi^2\alpha^2/M^2$, and therefore

$$T = C\exp\left(\frac{m^2\pi^2\alpha^2}{L^2}t + \frac{n^2\pi^2\alpha^2}{M^2}t\right).$$

Thus we obtain the solution

$$u(x,y,t) = F(x)G(y)T(t) = ABC\exp\left(\frac{m^2\pi^2\alpha^2}{L^2}t + \frac{n^2\pi^2\alpha^2}{M^2}t\right)\sin(m\pi x/L)\sin(n\pi x/M).$$

This motivates us to set

$$u_{mn}(x,y,t) = \exp\left(\frac{m^2\pi^2\alpha^2}{L^2}t + \frac{n^2\pi^2\alpha^2}{M^2}t\right)\sin(m\pi x/L)\sin(n\pi x/M).$$

Then by the superposition principle, any solution of the form

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}u_{mn}(x,y,t)$$

is a solution, for constants a_{mn}. In fact, all solutions may be written this way!

(c) From part (b), we know to try to write

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}u_{mn}(x,t)$$

for some constants a_{mn}, where here $\alpha^2 = 1, L = 1, M = 1$. we must choose the constants so that

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}u_{mn}(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin(m\pi x)\sin(n\pi x).$$

is equal to our initial condition. Looking at initial condition, this is easy! Just choose $a_{32} = 1, a_{24} = 1$ and $a_{mn} = 0$ otherwise. Thus:

$$u(x,y,t) = u_{32}(x,y,t) + u_{24}(x,y,t) = e^{-13\pi^2 t}\sin(3\pi x)\sin(2\pi y) + e^{-20\pi^2 t}\sin(2\pi x)\sin(4\pi y).$$

Plots at various times are included below:

$t = 0.000$ $t = 0.005$

$t = 0.010$ $t = 0.015$

Problem 2 *The Heat Equation in Polar Coordinates*

We consider the two dimensional heat equation

$$u_t - \alpha^2(u_{xx} + u_{yy}) = 0.$$

(a) Show that using polar coordinates, (r, θ), the heat equation becomes

$$u_t - \alpha^2 \left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \right) = 0.$$

(b) Assume that u is of the form $u(r, \theta, t) = R(r)S(\theta)T(t)$, and show that the heat equation reduces to the system of three ordinary differential equations

$$\begin{cases} T'(t) + \lambda T = 0 \\ r^2 R''(r) + rR'(r) + \frac{1}{\alpha^2}(r^2\lambda - \mu)R = 0 \\ S''(\theta) + \frac{\mu}{\alpha^2}S(\theta) = 0 \end{cases}$$

for some constants λ and μ.

(c) Explain why $\mu = n^2\alpha^2$ for some integer n. [Hint: remember that θ is the angle counter-clockwise from the x-axis].

(d) Find the general solution to the above system of equations in the case that $\lambda = 0$ and $\mu = \alpha^2$. [Hint: to solve for $R(r)$, propose a solution of the form $R(r) = r^b$]

. . . .

Solution 2.

(a) Note that $r^2 = x^2 + y^2$ and $\tan(\theta) = y/x$ and therefore

$$r_x = \cos(\theta), \quad r_y = \sin(\theta),$$

as well as

$$\theta_x = \frac{-\sin(\theta)}{r}, \quad \theta_y = \frac{\cos(\theta)}{r},$$

Then from the chain rule we have

$$\frac{\partial}{\partial x} = r_x \frac{\partial}{\partial r} + \theta_x \frac{\partial}{\partial \theta} = \cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = r_y \frac{\partial}{\partial r} + \theta_y \frac{\partial}{\partial \theta} = \sin(\theta)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}$$

Then we have that

$$\begin{aligned} \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial x}\right)^2 &= \left[\cos(\theta)\frac{\partial}{\partial r} + \frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right]^2 \\ &= \left[\cos(\theta)\frac{\partial}{\partial r}\right]^2 + \left[\cos(\theta)\frac{\partial}{\partial r}\right]\left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right] \\ &+ \left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right]\left[\cos(\theta)\frac{\partial}{\partial r}\right] + \left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right]^2 \end{aligned}$$

and also

$$\left[\cos(\theta)\frac{\partial}{\partial r}\right]^2 = \cos^2(\theta)\frac{\partial^2}{\partial r^2}.$$

$$\left[\cos(\theta)\frac{\partial}{\partial r}\right]\left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right] = \frac{-1}{r}\sin(\theta)\cos(\theta)\frac{\partial^2}{\partial r\partial \theta} + \frac{1}{r^2}\sin(\theta)\cos(\theta)\frac{\partial}{\partial \theta}$$

$$\left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right]\left[\cos(\theta)\frac{\partial}{\partial r}\right] = \frac{-1}{r}\sin(\theta)\cos(\theta)\frac{\partial^2}{\partial r\partial \theta} + \frac{1}{r}\sin^2(\theta)\frac{\partial}{\partial r}.$$

$$\left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right]^2 = \frac{1}{r^2}\sin^2(\theta)\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\sin(\theta)\cos(\theta)\frac{\partial}{\partial \theta}.$$

Similarly,

$$\begin{aligned}\frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial y}\right)^2 &= \left[\sin(\theta)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]^2 \\ &= \left[\sin(\theta)\frac{\partial}{\partial r}\right]^2 + \left[\sin(\theta)\frac{\partial}{\partial r}\right]\left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right] \\ &+ \left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]\left[\sin(\theta)\frac{\partial}{\partial r}\right] + \left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]^2\end{aligned}$$

and also

$$\left[\sin(\theta)\frac{\partial}{\partial r}\right]^2 = \sin^2(\theta)\frac{\partial^2}{\partial r^2}.$$

$$\left[\sin(\theta)\frac{\partial}{\partial r}\right]\left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right] = \frac{1}{r}\sin(\theta)\cos(\theta)\frac{\partial^2}{\partial r\partial \theta} - \frac{1}{r^2}\sin(\theta)\cos(\theta)\frac{\partial}{\partial \theta}.$$

$$\left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]\left[\sin(\theta)\frac{\partial}{\partial r}\right] = \frac{1}{r}\sin(\theta)\cos(\theta) + \frac{1}{r}\cos^2(\theta)\frac{\partial}{\partial r}.$$

$$\left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]^2 = \frac{1}{r^2}\cos^2(\theta)\frac{\partial^2}{\partial \theta^2} - \frac{1}{r^2}\sin(\theta)\cos(\theta)\frac{\partial}{\partial \theta}.$$

Adding all of this together, we obtain:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Applying this to the function u, we get

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Suubstiuting this into the heat equation leads to

$$u_t - \alpha^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = 0.$$

(b) Putting $u(r,\theta,t) = R(r)S(\theta)T(t)$ into the above expression, we obtain:

$$R(r)S(\theta)T'(t) = \alpha^2(R''(r)S(\theta)T(t) + \frac{1}{r}R'(r)S(\theta)T(t) + \frac{1}{r^2}R(r)S''(\theta)T(t)).$$

Dividing everything by $R(r)S(s)T(t)$, we obtain

$$T'(t)/T(t) = \alpha^2(R''(r)/R(r) + \frac{1}{r}R'(r)/R(r) + \frac{1}{r^2}S''(\theta)/S(\theta)).$$

The expression on the left hand side is a function of t only, while the right hand side is a function of r and θ only, and therefore both are equal to an arbitrary constant $-\lambda$. This leads to

$$T'(t)/T(t) = -\lambda$$

and also

$$\alpha^2(R''(r)/R(r) + \frac{1}{r}R'(r)/R(r) + \frac{1}{r^2}S''(\theta)/S(\theta)) = -\lambda.$$

Then after an algebraic manipulation,

$$\alpha^2 r^2 R''(r)/R(r) - \alpha^2 r R'(r)/R(r) - r^2\lambda = S''(\theta)/S(\theta).$$

Therefore again we have that both are equal to a constant $-\mu$:

$$\alpha^2 r^2 R''(r)/R(r) - \alpha^2 r R'(r)/R(r) - r^2\lambda = -\mu$$

and also

$$S''(\theta)/S(\theta) = -\mu.$$

After some simplification, this reduces to the expression in (b) above.

(c) Since (r,θ) and $(r,\theta+2\pi)$ both refer to the same coordinate in polar coordinates, the function S should satisfy $S(\theta) = S(\theta+2\pi)$. Moreover, we know that S satisfies $S''(\theta) + (\mu/\alpha^2)S(\theta) = 0$. Depending on the value of μ/α^2, the solutions to this are either exponentials, polynomials or trig. functions. Since we require S to be periodic, we want them to be trig functions, and therefore we need μ/α^2 to be positive. In this case, the general solution to the differential equation is

$$S(\theta) = A\cos(\sqrt{\mu}\theta/\alpha) + B\sin(\sqrt{\mu}\theta/\alpha).$$

Now we need S to be 2π periodic, and therefore we need $\sqrt{\mu}/\alpha = n$ for some integer n. Therefore $\mu = n^2\alpha^2$.

(d) Since $\lambda = 0$, the solution for T is $T = E$ for some constant E. Moreover, the general solution for S is

$$S(\theta) = A\cos(\theta) + B\sin(\theta)$$

Finally, we propose a solution $R(r) = r^b$ to the equation $r^2R''(r)+rR'(r)-R(r) = 0$, and therefore $b(b-1)r^{b-2}r^b + br^b - r^b = 0$. This leads to $b^2 - 1 = 0$, and therefore $b = \pm 1$. Thus we obtain two solutions: r and r^{-1}. The general solution is therefore

$$R(r) = Cr + Dr^{-1}$$

. Putting this all together, we obtain

$$u(r, \theta, t) = R(r)S(\theta)T(t) = (Cr + Dr^{-1})(A\cos(\theta) + B\sin(\theta))E.$$

Since A, B, C, D, E are all arbitrary constants, we can rewrite this as:

$$u(r, \theta, t) = C_1 r\cos(\theta) + C_2 r\sin(\theta) + C_3 r^{-1}\cos(\theta) + C_4 r^{-1}\sin(\theta).$$

Problem 3 *The Wave Equation I*

Consider an elastic string of length $L = 10$ whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x, 0) = f(x)$, and the material properties of the string make $u(x, t)$ satisfy the wave equation $u_{tt} - c^2 u_{xx}$ with $c = 1$. For each of the values of $f(x)$ below, determine

(i) Determine the solution $u(x, t)$ in terms of an infinite linear combination of the fundamental set of solutions $u_n(x, t) = \sin(n\pi x/L)\cos(cn\pi t/L)$

(ii) Plot $u(x, t)$ vs. x for $t = 0, 4, 8, 12, 16$

(iii) Describe the motion of the string in a few sentences.

(a)

$$f(x) = \begin{cases} 2x/L, & 0 \le x \le L/2 \\ 2(L - x), & L/2 < x \le L \end{cases}$$

(b)

$$f(x) = 8x(L - x)^2/L^3.$$

(c)

$$f(x) = \begin{cases} 1, & |x - L/2| < 1 \\ 0, & |x - L/2| \ge 1 \end{cases}$$

. . . .

Solution 3.

(a)

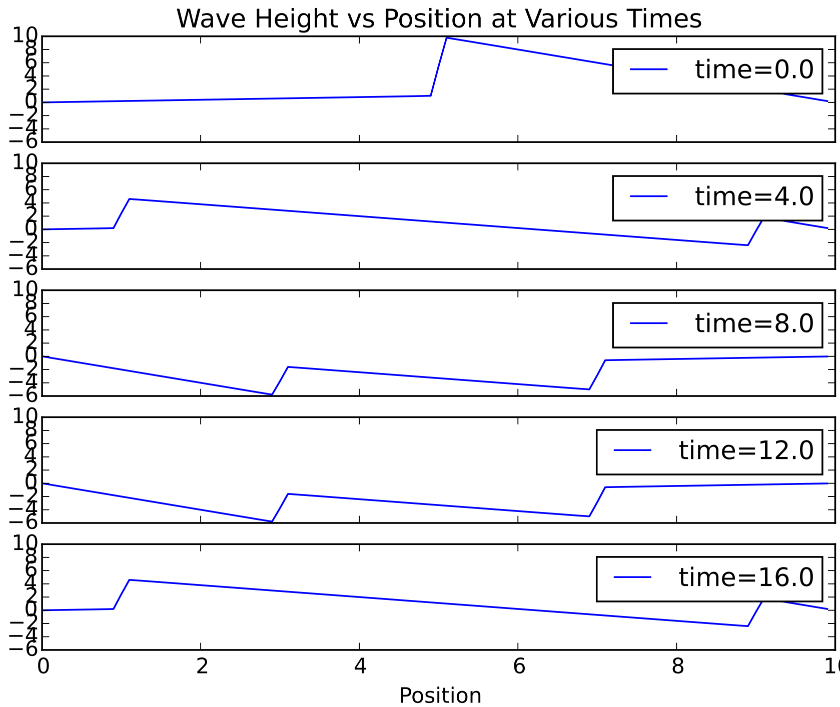
(i) The coefficients of the sine expansion of $f(x)$ are given by

$$a_n = \frac{4}{n^2\pi^2}\sin(n\pi/2) - \frac{2}{n\pi}\cos(n\pi/2) + \frac{4L}{n^2\pi^2}\sin(n\pi/2) + \frac{2L}{n\pi}\cos(n\pi/2)$$

The solution is then given by

$$u(x, n) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)\cos(cn\pi t/L).$$

(ii) The values of $u(x,t)$ at the specified times are shown in the graph below.



(iii) Initially, the string has a jagged jump at $x = 5$. As time goes on, this splits into two separate jagged pieces that move left and right, respectively. As they hit the ends of the string, each jagged part is reflected upside-down proceeding now in the opposite direction. As they pass through each other for the second time, they temporarily form a single jagged piece like the original, but now upside-down. This immediately breaks apart again into the two jagged pieces, which are again reflected when they reach the end of the string, and finally arrive back in the center, forming once again the original state of the string.

(b)

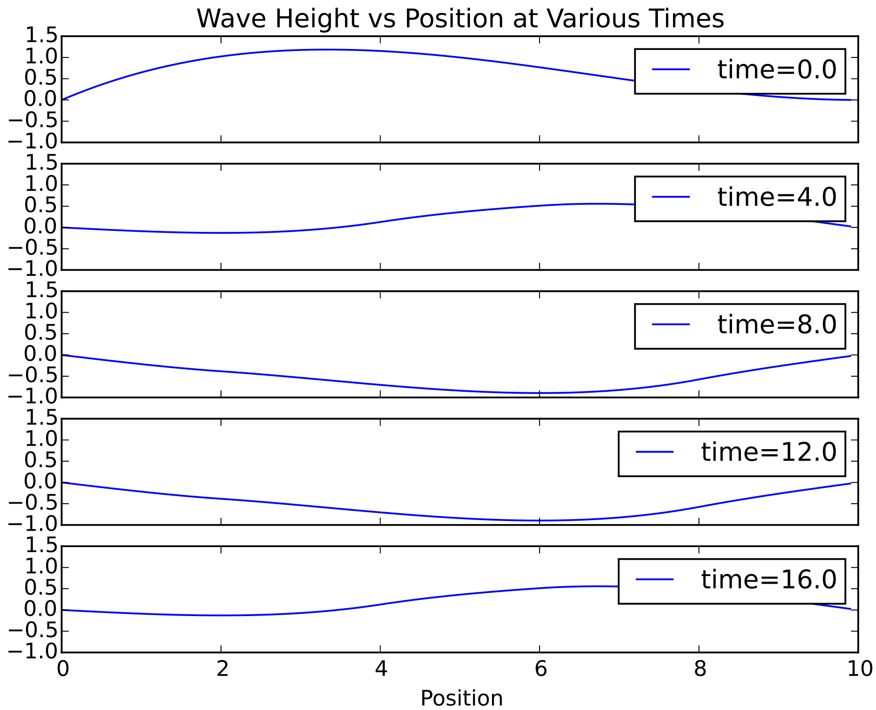
(a) The coefficients of the sine expansion of $f(x)$ are given by

$$a_n = \frac{64}{n^3\pi^3} + (-1)^n \frac{32}{n^3\pi^3}.$$

The solution is then given by

$$u(x,n) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)\cos(cn\pi t/L).$$

(b) The values of $u(x,t)$ at the specified times are shown in the graph below.



(c) Initially, the string has a smooth hump slightly skewed to the left. This hump moves off to the right, and when it reaches the end of the string, it is flipped upside down, and then moves to the left. When it next reaches the other end of the string it flips again, and moves back to the right, eventually returning to the initial state.

(c)

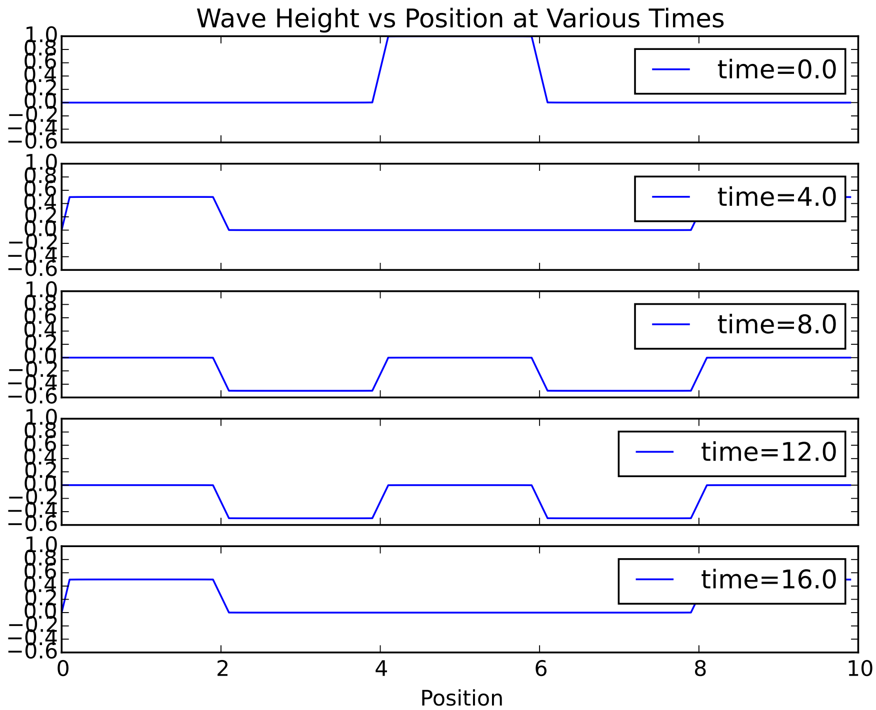
(a) The coefficients of the sine expansion of $f(x)$ are given by

$$a_n = \frac{2}{n\pi}(\cos(2n\pi/5) - \cos(3n\pi/5)).$$

The solution is then given by

$$u(x, n) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L) \cos(cn\pi t/L).$$

(b) The values of $u(x, t)$ at the specified times are shown in the graph below.



(c) Initially, the string has a square bump in the center. This splits up into two square bumps, moving in opposite directions. When the bumps reach the ends, they are reflected, so that they are upside down and moving in the opposite directions they were before. As they pass over each other, they reform the original square bump, but upside down. This then splits up and the two squares move again to opposite ends of the string, where they are again reflected. These eventually meet again in the middle, returning the string to the original state.

Problem 4 *The Wave Equation II*

Consider an elastic string of length $L = 10$ whose ends are held fixed. The string is set in motion from its equilibrium position with initial velocity given by $u_t(x, 0) = g(x)$, and the material properties of the string make $u(x, t)$ satisfy the wave equation $u_{tt} - c^2 u_{xx}$ with $c = 1$. For each of the values of $g(x)$ below, determine

(i) Determine the solution $u(x, t)$ for $0 \le x \le L$ and $t > 0$ in terms of an infinite linear combination of the fundamental set of solutions $u_n(x, t) = \sin(n\pi x/L)\sin(cn\pi t/L)$

(ii) Plot $u(x, t)$ vs. x for $t = 0, 4, 8, 12, 16$

(iii) Describe the motion of the string in a few sentences.

(a)

$$g(x) = \begin{cases} 2x/L, & 0 \le x \le L/2 \\ 2(L - x), & L/2 < x \le L \end{cases}$$

(b)

$$g(x) = 8x(L-x)^2/L^3.$$

(c)

$$g(x) = \begin{cases} 1, & |x - L/2| < 1 \\ 0, & |x - L/2| \geq 1 \end{cases}$$

. . . .

Solution 4.

(a)

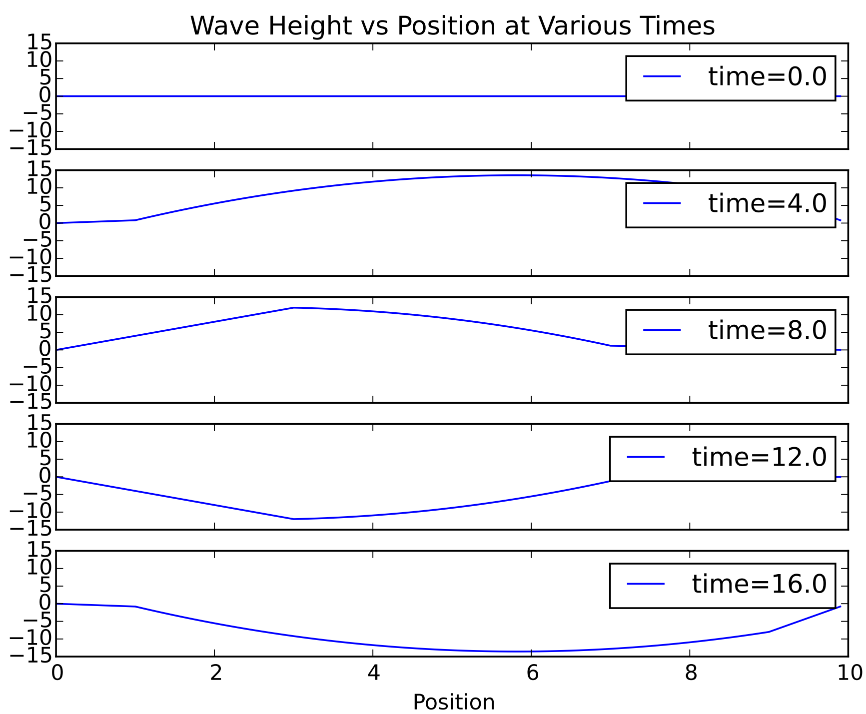
(i) The coefficients of the sine expansion of $f(x)$ are given by

$$a_n = \frac{4}{n^2\pi^2}\sin(n\pi/2) - \frac{2}{n\pi}\cos(n\pi/2) + \frac{4L}{n^2\pi^2}\sin(n\pi/2) + \frac{2L}{n\pi}\cos(n\pi/2)$$

The solution is then given by

$$u(x,n) = \sum_{n=1}^{\infty} \frac{a_n L}{cn\pi}\sin(n\pi x/L)\sin(cn\pi t/L).$$

(ii) The values of $u(x,t)$ at the specified times are shown in the graph below.



(iii) Description is left to the reader.

(b)

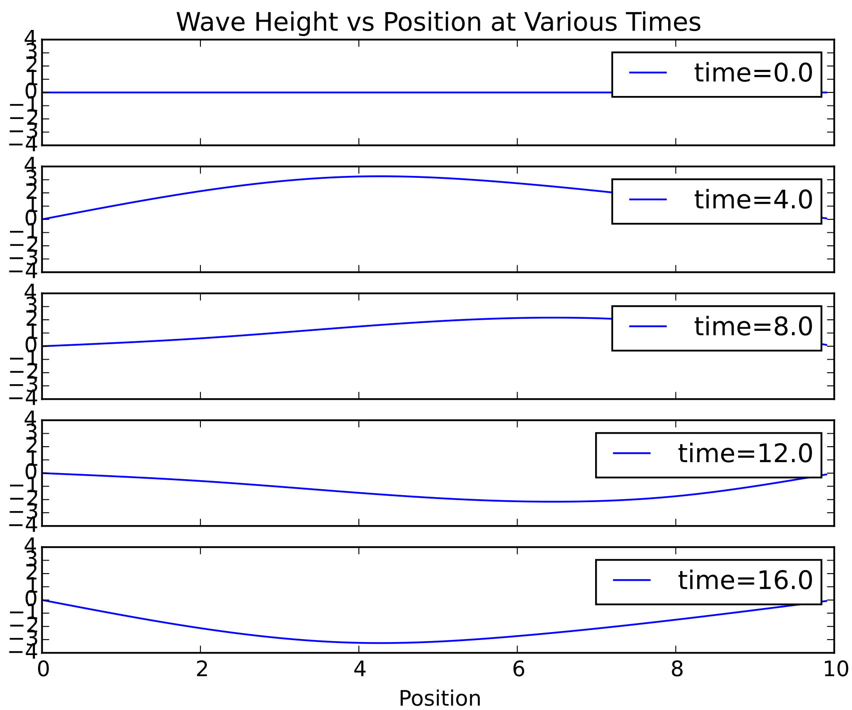
(i) The coefficients of the sine expansion of $f(x)$ are given by

$$a_n = \frac{64}{n^3\pi^3} + (-1)^n \frac{32}{n^3\pi^3}.$$

The solution is then given by

$$u(x,n) = \sum_{n=1}^{\infty} \frac{a_n L}{cn\pi} \sin(n\pi x/L)\sin(cn\pi t/L).$$

(ii) The values of $u(x,t)$ at the specified times are shown in the graph below.



(iii) Description is left to the reader.

(c)

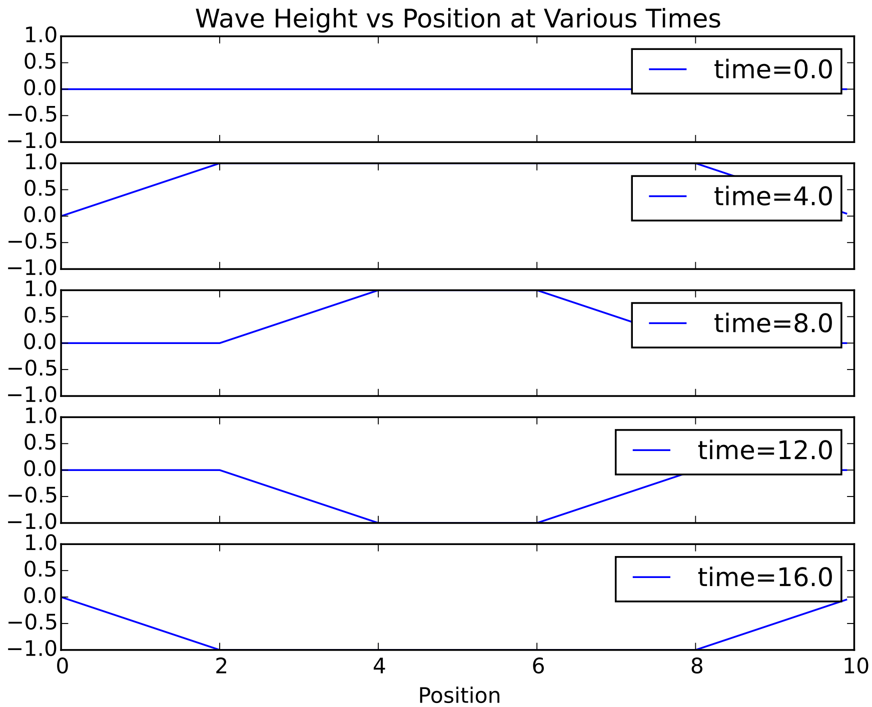
(i) The coefficients of the sine expansion of $f(x)$ are given by

$$a_n = \frac{2}{n\pi}(\cos(2n\pi/5) - \cos(3n\pi/5)).$$

The solution is then given by

$$u(x,n) = \sum_{n=1}^{\infty} \frac{a_n L}{cn\pi} \sin(n\pi x/L)\sin(cn\pi t/L).$$

(ii) The values of $u(x,t)$ at the specified times are shown in the graph below.



(iii) Description is left to the reader.

Problem 5 Some Physics Flavor

A steel wire 5 ft in length is stretched by a tensile force of 50 lb. The wire has a weight per unit lenght of 0.026 lb/ft.

(a) Find the velocity of propagation of transverse waves in the wire.

(b) Find the natrual frequencies of vibration.

(c) If the tension in the wire is increased, how are the natural frequencies changed? Are the natural modes also changed?

. . . .

Solution 5.

(a) The velocity of propagation is $c = \sqrt{T/\rho}$, where T is the tension and ρ is the density. The tension is 50 lbs, and the weight per unit length is 0.026 lb/ft. Using a gravitational acceleration of $g = 32$ ft/s^2, we have that the mass per unit length (eg. density) is $\rho = 0.026/32$ (in units of slugs, eg. lb·s^2/ft^2. Thus $c = \sqrt{(50 * 32/0.026)} = 248.069$ ft/s.

(b) The (time) periods of oscillation of various modes are given by $T = 2L/(nc)$, and the corresponding (angular) frequencies are therefore $f = 2\pi/T = \pi nc/(L) = 49.6138n\pi$.

(c) If we increase the tension of the wire, then c increases and as we see this increases the values of the natural frequencies. The natural modes remain unchanged, since the modes depend only on the length L of the string.

Problem 6 D'Alembert's Method

Use D'Alembert's Method to find a solution to the wave equation

$$u_{tt} - u_{xx} = 0, \quad 0 \leq x \leq 1, \ t > 0$$

satisfying $u(0) = 0$ and $u(1) = 0$, with the property that $u(x, 0) = \sin^3(\pi x)$. Use this solution to create a surface plot of $u(x, t)$ for $0 \leq x \leq 1$ and $0 \leq t \leq 4$.

. . . .

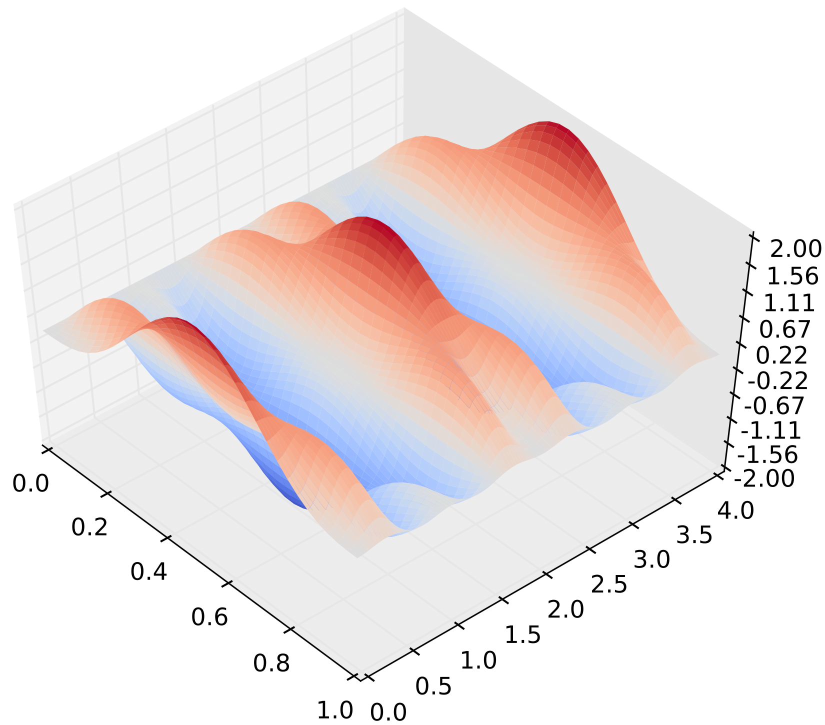
Solution 6. D'Alembert says that if we take our original $u(x, 0)$, which is defined for $0 \leq x \leq L$, and then reflect it oddly in the interval $-L \leq x \leq L$, and then extend it $2L$-periodically, to obtain a function $f(x)$, then a solution to our question is given by

$$u(x, t) = \frac{1}{2}(f(x - ct) + f(x + ct)).$$

Now since $\sin^3(\pi x)$ is already odd with the right period, we can just take $f(x) = \sin^3(\pi x)$ in this case! Therefore the solution we want is

$$u(x, t) = \frac{1}{2}(\sin^3(\pi(x - ct)) + \sin^3(\pi(x + ct))).$$

A surface plot of our solution is included below.



Problem 7 Wave Equation with von Neumann Boundary Conditions

Use separation of variables to find a solution to the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

with the homogeneous von Neumman boundary conditions

$$u_x(0,t) = 0, \quad u_x(L,t) = 0,$$

and satisfying the initial condition

$$u(x,0) = \cos(n\pi x/L), u_t(x,0) = 0,$$

where here n is a nonnegative integer.

. . . .

Solution 7. For separation of variables, we propose a solution $u(x,t) = F(x)T(t)$. Then the usual work reduces to the equations

$$F''(x) + \frac{\lambda}{c^2}F(x) = 0, \quad T''(t) + \lambda T(t) = 0.$$

The interesting thing is how the Neumann boundary conditions induce Neumann boundary conditions on $F(x)$. Since $u_x(0,t) = 0$, we see that $F'(0)G(t) = 0$. Since $G(t)$ cannot be zero for a nontrivial solution, we must hvae $F'(0) = 0$. Similarly, we see that $F'(L) = 0$. Therefore we have the homogeneous Neumann boundary value problem

$$F''(x) + \frac{\lambda}{c^2}F(x) = 0,\ F'(0) = 0,\ F'(L) = 0.$$

The usual considerations show that $\lambda = n^2\pi^2c^2/L^2$, and that $F(x) = C\cos(\sqrt{\lambda}x/c)$ for some constant C. Furthermore, since $u_t(x,0) = 0$, we see that $G'(0) = 0$, and therefore $G(t) = B\cos(\sqrt{\lambda}t)$. Therefore we see that

$$u(x,t) = F(x)G(t) = BC\cos(\sqrt{\lambda}x)\cos(\sqrt{\lambda}t) = BC\cos(n\pi x/L)\cos(cn\pi t/L).$$

To satisfy the initial condition, we take $BC = 1$, and therefore

$$u(x,t) = \cos(n\pi x/L)\cos(cn\pi t/L).$$