Math 309 Lecture 10

More Jordan Normal Form

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Today!

Plan for today:

- 3×3 Jordan Normal Form
- Calculating a Fundamental Matrix

Next time:

- Nonhomogeneous Differential Equations

Outline

1. 3×3 Jordan Normal Form
   - Calculating the Normal Form
2. Calculating a Fundamental Matrix
   - Exponentials of Jordan Blocks
   - Examples

Possible Jordan Normal Forms

- Four possible Jordan normal forms:

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \quad \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

- the two cases with a 1×1 and a 2×2 Jordan block are considered equivalent

Obtaining Jordan Normal Form

- any 3×3 matrix A has a Jordan normal form N – unique up to rearranging order of blocks
- ie. there exists an invertible matrix P with $P^{-1}AP = N$
- CAREFUL! P not unique!!
- blocks and sizes are determined by eigenvalues of A and their multiplicities!
- four possibilities depending on eigenvlaues and degeneracy

POSSIBILITY 1

- *A* is nondegenerate with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (not necessarily distinct!)
- then the Jordan normal form for *A* is

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

POSSIBILITY 2

- A is degenerate with eigenvalues $\lambda_1, \lambda_2, \lambda_2$
- degeneracy implies λ_2 has geom. mult 1
- then the Jordan normal form for A is

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

POSSIBILITY 3

- A is degenerate with eigenvalues $\lambda_1, \lambda_1, \lambda_1$
- degeneracy implies λ_1 has geom. mult 1 or 2
- if geom mult. is 2 then then the Jordan normal form for A is

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

POSSIBILITY 4

- A is degenerate with eigenvalues $\lambda_1, \lambda_1, \lambda_1$
- degeneracy implies λ_1 has geom. mult 1 or 2
- if geom mult. is 1 then then the Jordan normal form for A is

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

Calcualting P

- now we know how to determine the Jordan normal form of A from its eigenvalues and multiplicities
- next obvious question is:

Question

Suppose A is a 3×3 matrix with Jordan normal form N. How do we find P so that $P^{-1}AP = N$?

- we will explain for each of the four different possibilities separately

POSSIBILITY 1

- in this case A is nondegenerate, hence diagonalizable
- we calculate eigenbases for each of the eigenspaces of A
- use these eigenbases as the columns of P
- Jordan normal form is diagonal matrix
- order of the basis elements determines order of $\lambda_1, \lambda_2, \lambda_3$ in diagonal matrix

POSSIBILITY 2

- in this case A has two eigenvalues λ_1, λ_2
- λ_1 has alg. mult 1, geom. mult 1
- λ_2 has alg. mult 2, geom. mult 1
- STEPS TO FIND P:

STEP 1: choose $\vec{u} \in E_{\lambda_1}(A)$
STEP 2: choose $\vec{v} \in E_{\lambda_2}(A)$
STEP 3: find a solution $\vec{w}$ to $(A - \lambda I)\vec{w} = \vec{v}$
STEP 4: take $P = [\vec{u}\ \vec{v}\ \vec{w}]$ (order is important!)

- then we have

$$P^{-1}AP = N, \quad N = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

POSSIBILITY 3

- in this case A has exactly one eigenvalue λ_1
- λ_1 has alg. mult 3, geom. mult 2
- STEPS TO FIND P:

STEP 1: choose $\vec{w} \notin E_{\lambda_1}(A)$
STEP 2: set $\vec{v} = (A - \lambda I)\vec{w}$
STEP 3: find a basis $\vec{u}_1, \vec{u}_2$ for $E_{\lambda_1}(A)$
STEP 4: if $\vec{v} \notin \text{span}(\vec{u}_1)$, choose $\vec{u} = \vec{u}_1$; otherwise take $\vec{u} = \vec{u}_2$
STEP 5: take $P = [\vec{u}\ \vec{v}\ \vec{w}]$ (order is important!)

- then we have

$$P^{-1}AP = N, \quad N = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

POSSIBILITY 4

- in this case A has exactly one eigenvalue λ_1
- λ_1 has alg. mult 3, geom. mult 1
- STEPS TO FIND P:

STEP 1: choose $\vec{w} \notin E_{\lambda_1}(A)$
STEP 2: set $\vec{v} = (A - \lambda I)\vec{w}$
STEP 3: set $\vec{u} = (A - \lambda I)\vec{v}$
STEP 4: if $\vec{u} = \vec{0}$, pick a different w and start over with STEP 1, othewise continue to STEP 5
STEP 5: take $P = [\vec{u}\ \vec{v}\ \vec{w}]$ (order is important!)

- then we have

$$P^{-1}AP = N, \quad N = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

Review

- recall that a fundamental matrix of

$$\vec{y}'(t) = A\vec{y}(t)$$

- is given by

$$\Psi(t) = \exp(At)$$

- so calculating a fundamental matrix is only as hard as calculating a matrix exponential
- we calculate matrix exponentials by Jordan normal form!

- the following Prop. shows how the exponential of a matrix is related to the exponential of its Jordan normal form

Proposition

If $P^{-1}AP = N$, then $\exp(At) = P\exp(Nt)P^{-1}$.

- this is useful, because calculating the exponential of a matrix in Jordan normal form is easy!

Exponentials of 2×2 Jordan normal forms

- There are two possible forms for a 2×2 matrix in Jordan normal form:

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad N = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}.$$

- Direct calculation shows

$$\exp(Dt) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

- Moreover clever calculation shows

$$\exp(Nt) = \begin{pmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{pmatrix}$$

Exponentials of 3 × 3 Jordan normal forms

- There are four possible forms for a 3 × 3 Jordan normal matrix
- the first two have exponential forms given by:

$$N = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \implies \exp(Nt) = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix}$$

$$N = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix} \implies \exp(Nt) = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & e^{\lambda_2 t} \end{pmatrix}$$

Exponentials of 3×3 Jordan normal forms

- There are four possible forms for a 3×3 Jordan normal matrix
- the secont two have exponential forms given by:

$$N = \begin{pmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} \implies \exp(Nt) = \begin{pmatrix} e^{\lambda_2 t} & te^{\lambda_2 t} & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_1 t} \end{pmatrix}$$

$$N = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \implies \exp(Nt) = \begin{pmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2 e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{pmatrix}$$

Example 1

Question

Find a fundamental matrix for the equation

$$y' = Ay, \quad A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

- we diagonalize: $P^{-1}AP = D$ for

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Psi(t) = \exp(At) = P\exp(Dt)P^{-1}$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+e^{2t} & 1-e^{2t} \\ 1-e^{2t} & 1+e^{2t} \end{pmatrix}$$

Example 2

Question

Find a fundamental matrix for the equation

$$y' = Ay, \quad A = \begin{pmatrix} 7 & 1 \\ -1 & 5 \end{pmatrix}$$

- A is not diagonalizable!! (oh no)
- how do we calculate e^{At}?
- we can put it into Jordan normal form
- how do we calculate matrix exponential of a Jordan block?

Exponentials of Jordan Blocks

Question

Waht is $\exp(J_m(\lambda)t)$?

$$\exp(J_2(\lambda)t) = \exp(It)\exp(j_2(\lambda)t - It) = \exp(It)\exp\left(\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\right)$$

$$= \exp(It)\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

- exponentials of larger Jordan blocks work similarly

Example 2

Question

Find a fundamental matrix for the equation

$$y' = Ay, \quad A = \begin{pmatrix} 7 & 1 \\ -1 & 5 \end{pmatrix}$$

- take

$$P = \begin{pmatrix} 1 & 1/2 \\ -1 & 1/2 \end{pmatrix} \quad D = \begin{pmatrix} 6 & 1 \\ 0 & 6 \end{pmatrix}$$

- therefore

$$\Psi(t) = \exp(At) = P\exp(J_2(6)t)P^{-1}$$

- we can calculate this now...

summary!

what we did today:

- diagonalizable matrices
- jordan normal form
- calculating a fundamental matrix

plan for next time:

- nonhomogeneous differential equations