Math 309 Lecture 5

Constant Coefficient Homogeneous Linear Systems of ODEs

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Today!

Plan for today:

- Direction Fields

Next time:

- Complex Eigenvalues
- Stability of the Origin

Outline

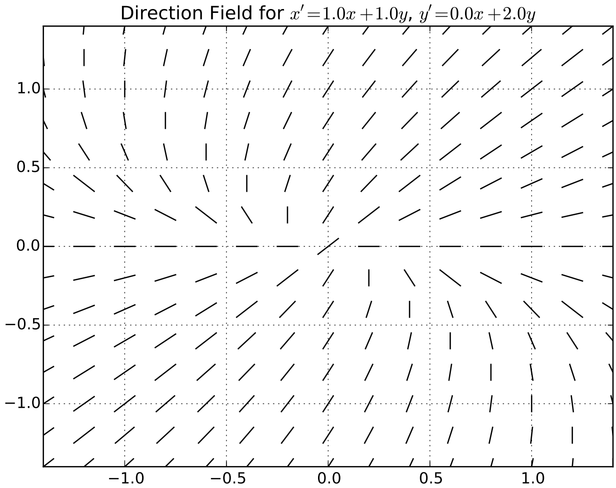
Direction Fields

Consider a 2×2 system

$$\begin{aligned} y_1' &= ay_1 + by_2 \\ y_2' &= cy_1 + dy_2 \end{aligned}$$

- solving this system has both algebraic and geometric interpretations
- we can draw a “picture" of the equation in the **phase plane**
- here by **phase plane** we mean the y_1, y_2 plane
- strategy: at each point (y_1, y_2) draw a dash in direction of vector
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix}.$$
- result is called a **direction field**

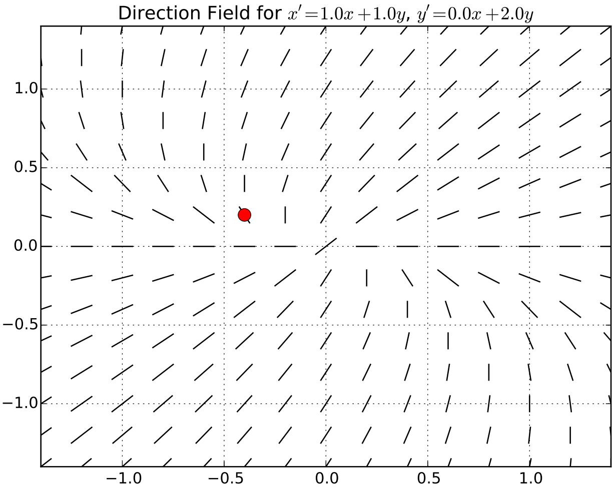
Example Direction Field



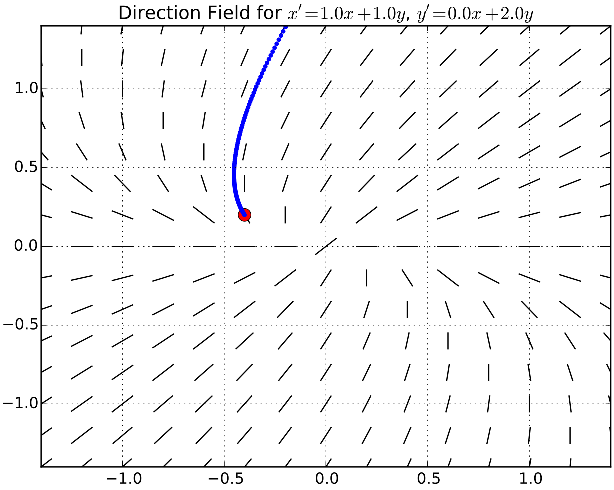
Solutions as Tangent Curves

- think of slope field as current in the ocean
- solutions to the system of equation are traced out by path of a (slow) boat
- the path a boat takes traces a curve whose tangent lines always point in direction of local slope field

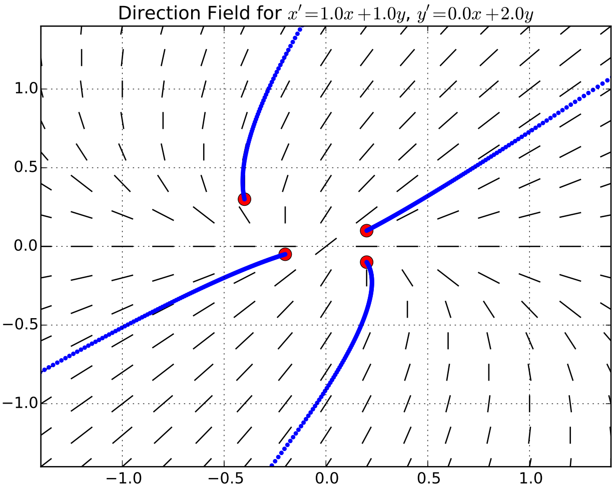
A Boat in the Ocean



The Path of the Boat



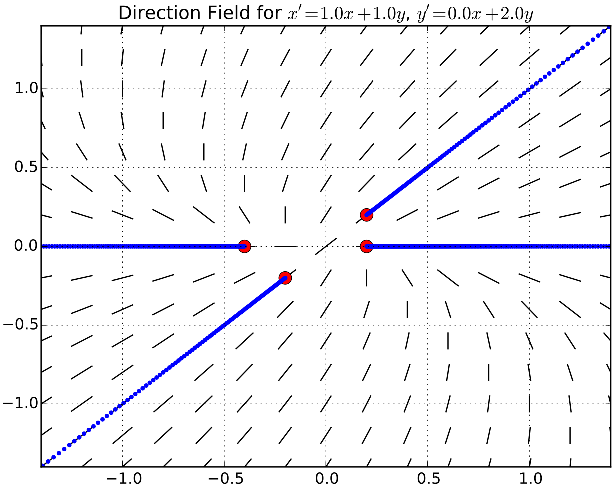
More Possible Paths



Observations:

- regardless of the initial position, the “boat" moves away from the origin
- unless if the boat starts at the origin, in which case it stays there
- for this reason, in this case we call the origin an **exponentially unstable node**
- note that there are also two straight paths the boat can take – corresponding to eigenvectors!

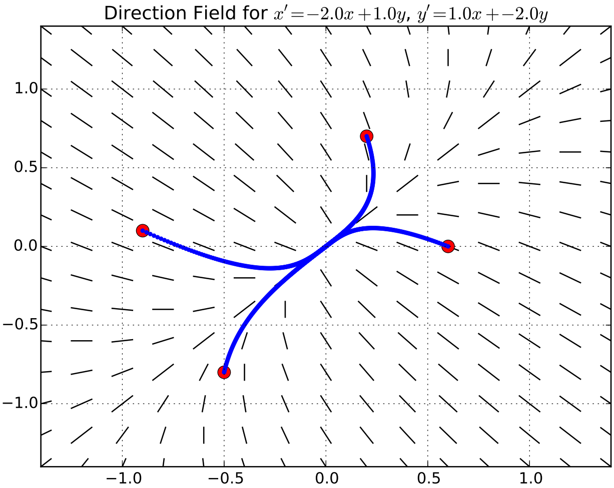
Straight Paths from Eigenvectors



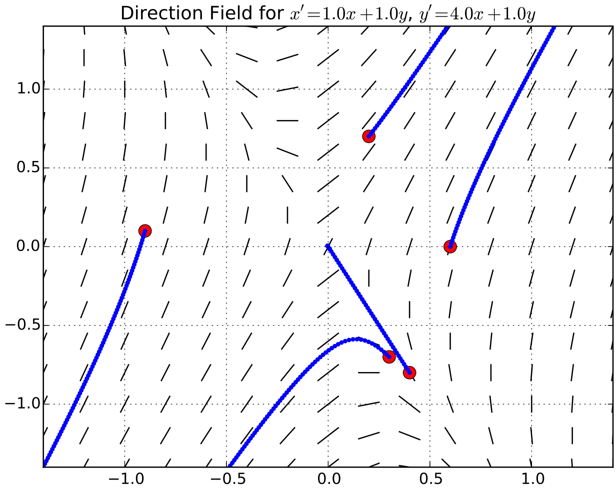
Saddle Point or Node

- the origin does not have to be an exponentially unstable node
- it may also be a **exponentially stable node** or a **saddle point**
- for an exponentially stable node, solutions tend toward the origin
- for a saddle point, solutions tend both toward and away from the origin, based on the initial condition
- for the equation $\vec{y}'(t) = A\vec{y}(t)$, the behavior of solutions around the origin depends on the *eigenvalues* of A

Exponentially Stable Node



Saddle Node


Direction Field for $x' = 1.0x + 1.0y$, $y' = 4.0x + 1.0y$
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

Behavior of the Origin

- the origin is *always* a fixed point of $\vec{y}'(t) = Ay(t)$
- eg. $\vec{y}(t) = \vec{0}$ is a constant solution of the equation
- how other solutions behave is based on the eigenvalues of A:

(a) if both eigenvalues of A are real and positive, then origin is an exponentially unstable node

(b) if both eigenvalues of A are real and negative, then origin is an exponentially stable node

(c) if both eigenvalues of A are mixed sign, then origin is a saddle point

(d) what about when the eigenvalues of A are complex?

Summary!

What we did today:

- Direction Fields

Plan for next time:

- Complex Eigenvalues
- Stability of the Origin