Math 309 Lecture 6 and 7

Constant Coefficient Homogeneous Linear Systems of ODEs

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Today!

Plan for today:

- Complex Eigenvalues
- Stability of the Origin

Next time:

- Fundamental Matrix

Outline

1. Complex Eigenvalues and Slope Fields
   - Slope Fields for Complex Eigenvalues
   - General Solution
2. Stability of the Origin
   - Stability and Eigenvalues

Review of Results from Last Time

- the origin is *always* a fixed point of $\vec{y}'(t) = Ay(t)$
- eg. $\vec{y}(t) = \vec{0}$ is a constant solution of the equation
- how other solutions behave is based on the eigenvalues of A:

(a) if both eigenvalues of A are real and positive, then origin is an exponentially unstable node

(b) if both eigenvalues of A are real and negative, then origin is an exponentially stable node

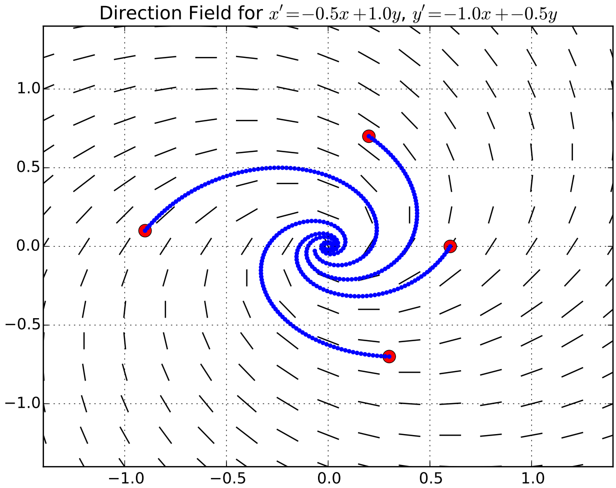
(c) if both eigenvalues of A are mixed sign, then origin is a saddle point

(d) what about when the eigenvalues of A are complex?

Spirally Slope Fields

- slope fields for complex eigenvalues are characterized by spiral patterns
- for example:
$$A = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}$$
- characteristic polynomial is
$$p_A(x) = \det(A - xI) = x^2 + x + \frac{5}{4}$$
- eigenvalues of A are $-(1/2) \pm i$

Complex Eigenvalues: $-(1/2) \pm i$

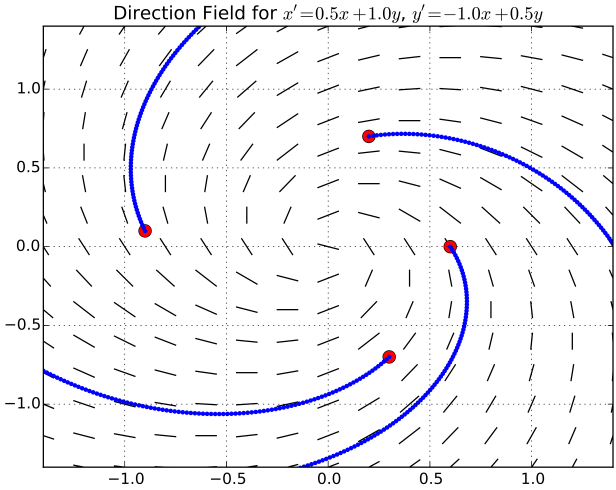


Behavior of the Origin

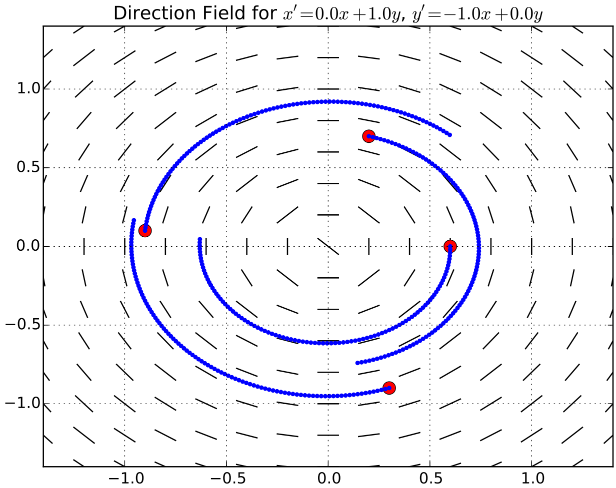
- suppose that A has complex eigenvalues
- they come in conjugate pairs! $\lambda_1 = a + ib$, $\lambda_2 = a - ib$
- the origin is *always* a fixed point of $\vec{y}'(t) = Ay(t)$
- whether our ship moves toward or away depends on value of a

(a) if a is positive, move away
(b) if a is negative, move toward
(c) if a is zero, circle around

Complex Eigenvalues: $(1/2) \pm i$



Complex Eigenvalues: $\pm i$



What about General Solutions?

Question

How do we find the general solution in the case that A has complex eigenvalues?

- use Euler's definition!

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

- we can then take our eigenvalue solutions and write them as linear combinations of *real* solutions

Example

Question

Find the general solution of the equation

$$\vec{y}'(x) = A\vec{y}, \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

- first we find the eigenvalues: $-(1/2) \pm i$
- then we find the corresponding eigenspaces:

$$E_{-(1/2)+i} = \text{span}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\} \quad E_{-(1/2)-i} = \text{span}\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\}$$

Example

- from this we get two (complex) solutions

$$\vec{y}_1(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t} \quad \vec{y}_2(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t}$$

- by the superposition principal we get the family of solutions:

$$\begin{aligned}
\vec{y}(t) &= c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t} \\
&= c_1 e^{-t/2} \begin{pmatrix} \cos(t) + i\sin(t) \\ i\cos(t) - \sin(t) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \cos(t) - i\sin(t) \\ -i\cos(t) - \sin(t) \end{pmatrix} \\
&= e^{-t/2} \begin{pmatrix} (c_1 + c_2)\cos(t) + i(c_1 - c_2)\sin(t) \\ i(c_1 - c_2)\cos(t) - (c_1 + c_2)\sin(t) \end{pmatrix} \\
&= e^{-t/2} \begin{pmatrix} b_1\cos(t) + b_2\sin(t) \\ b_2\cos(t) - b_1\sin(t) \end{pmatrix} \\
&= b_1 e^{-t/2} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + b_2 e^{-t/2} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}
\end{aligned}$$

Correspondence

- the stability of the origin depends on the eigenvalues of the matrix A
- five possibilities:
  (1) all eigenvalues are real and negative –> origin is exponentially stable (sink)
  (2) all of the eigenvalues are real and positive –> origin is unstable (source)
  (3) eigenvalues are real and opposite-signed –> origin is saddle
  (4) both of the eigenvalues are complex with positive real component –> origin is spirally unstable (spiral source)
  (5) both of the eigenvalues are complex with nonpositive real component –> origin is spirally stable (spiral sink)

Alternative Correspondence

- the stability may also be classified based on the determinant and trace of A
- this is because

$$p_A(x) = x^2 - \text{tr}(A)x + \det(A).$$

- five possibilities:
  (1) $\text{tr}(A)^2 > 4\det(A)$ and $\det(A) < 0$ –> saddle
  (2) $\text{tr}(A)^2 > 4\det(A)$, $\det(A) > 0$, and $\text{tr}(A) > 0$ –> unstable
  (3) $\text{tr}(A)^2 > 4\det(A)$, $\det(A) > 0$, and $\text{tr}(A) < 0$ –> exponentially stable
  (4) $\text{tr}(A)^2 < 4\det(A)$ and $\text{tr}(A) \leq 0$ –> spirally stable
  (5) $\text{tr}(A)^2 < 4\det(A)$ and $\text{tr}(A) > 0$ –> spirally unstable

Stability Picture

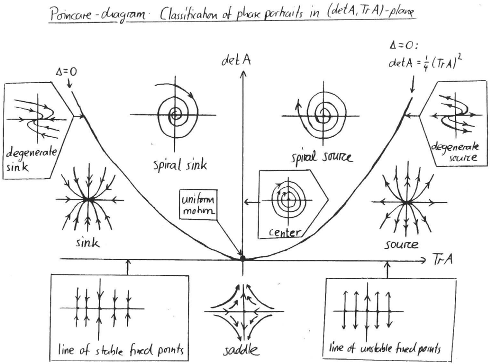


Figure: Picture of Classification (DR Hundley, Whitman College)

Summary!

What we did today:

- Complex Eigenvalues
- Stability of the Origin

Plan for next time:

- Fundamental Matrix