Math 307 Quiz 1

April 7, 2016

Problem 1. Suppose that A is an $n \times n$ matrix, and that $\vec{v}$ is an eigenvector of A with eigenvalue λ. Show that $y = e^{\lambda x}\vec{v}$ is a solution to the differential equation

$$\frac{d}{dx}\vec{y} = A\vec{y}.$$

Solution 1. We calculate

$$\begin{aligned}
\frac{d}{dx}\vec{y} &= \frac{d}{dx}(e^{\lambda x}\vec{v}) \\
&= \left(\frac{d}{dx}e^{\lambda x}\right)\vec{v}) \qquad \text{since } \vec{v} \text{ is constant} \\
&= (\lambda e^{\lambda x})\vec{v} \\
&= e^{\lambda x}(\lambda\vec{v}) \\
&= e^{\lambda x}(A\vec{v}) \qquad \text{since } \vec{v} \text{ is an eigenvector with eigenvalue } \lambda \\
&= A(e^{\lambda x}\vec{v}) = A\vec{y}.
\end{aligned}$$

This shows that $\frac{d}{dx}\vec{y} = A\vec{y}$.

Problem 2. Find the general solution of the differential equation

$$\frac{d}{dx}\vec{y} = A\vec{y}, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Solution 2. The characteristic polynomial is

$$p_A(x) = (2-x)^2 - 1 = x^2 - 4x + 3 = (x-3)(x-1).$$

This says that A has eigenvalues 1 and 3 with both with algebraic multiplicity one, and hence both with geometric multiplicity one. The corresponding eigenspaces are

$$E_1(A) = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\} \quad E_3(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

Therefore the general solution is

$$\vec{y}(x) = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^x + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3x}.$$