Math 309 Quiz 2

April 15, 2016

Problem 1. Consider the system of differential equations

$$\frac{d}{dx}y = A\vec{y}, \quad A = \begin{pmatrix} a & a \\ 1 & 1-a \end{pmatrix}.$$

Here a is a real number.

(a) For which values of a is the origin a source (unstable node)?

(b) For which values of a is the origin a sink (asymptotically stable node)?

(c) For which values of a is the origin a saddle point?

Solution 1. We calculate that $p_A(x) = x^2 - x - a^2$, and therefore that the eigenvalues of a are $(1 \pm \sqrt{1+4a^2})/2$.

1. The origin is a source if both of the eigenvalues are positive. This happens when $1 > \sqrt{1+4a^2}$, which is never true.
2. The origin is a sink if both eigenvalues are negative. Since $1+\sqrt{1+4a^2}$ is a positive eigenvalue, this never happens either.
3. The origin is a saddle point if one of the eigenvalues is positive, and the other is negative. This happens for all $a \neq 0$.

Problem 2. Find the (real) general solution of the differential equation

$$\frac{d}{dx}\vec{y} = A\vec{y}, \quad A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Solution 2. We calculate $p_A(x) = x^2 + 2x + 2$, and therefore the eigenvalues are $-1 \pm i$. We calculate the corresponding eigenspaces to be

$$E_{-1+i} = \operatorname{span}\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}, \quad E_{-1-i} = \operatorname{span}\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$$

The eigenvectors and eigenvalues give us two linearly independent solutions $\vec{y}_1, \vec{y}_2$. Using the superposition principal, the (complex) general solution is therefore

$$\vec{y} = c_1 \overbrace{e^{(-1+i)x} \begin{pmatrix} -i \\ 1 \end{pmatrix}}^{\vec{y}_1} + c_2 \overbrace{e^{(-1-i)x} \begin{pmatrix} i \\ 1 \end{pmatrix}}^{\vec{y}_2}.$$

However, we wanted the real general solution! To get this, we take one of the solutions we found above, (say $\vec{y}_1$) and instead use its real and imaginary parts. We calculate

$$\begin{aligned}
\vec{y}_1 &= e^{(-1+i)x}\begin{pmatrix} -i \\ 1 \end{pmatrix} \\
&= e^{-x}\begin{pmatrix} -ie^{ix} \\ e^{ix} \end{pmatrix} = e^{-x}\begin{pmatrix} \sin(x) - i\cos(x) \\ \cos(x) + i\sin(x) \end{pmatrix} \\
&= e^{-x}\begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix} + ie^{-x}\begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix}.
\end{aligned}$$

We proved in class that the real part and imaginary part of a solution are themselves solutions and thus we get two real solutions:

$$\mathrm{Re}(\vec{y}_1) = e^{-x}\begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix}, \quad \mathrm{Im}(\vec{y}_1) = e^{-x}\begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix}.$$

THen using the superposition principal, we get the real general solution

$$\vec{y} = b_1 e^{-x}\begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix} + b_2 e^{-x}\begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix}.$$