Math 309 Quiz 3

May 25, 2016

Problem 1. Suppose that A is a 3×3 matrix with a single eigenvalue λ of algebraic multiplicity 3 and geometric multiplicity 2. What is the Jordan normal form of A?

Solution 1. The Jordan normal form of A is $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ (remember, these are actually equivalent!).

Problem 2. Calculate a fundamental matrix for the system of equations

$$\frac{d}{dx}\vec{y} = A\vec{y}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution 2. The eigenvalues of A are 0 and 2, and associated eigenvectors are $\binom{1}{-1}$ and $\binom{1}{1}$. Therefore a fundamental matrix is

$$\Psi(x) = \begin{pmatrix} 1 & e^{2x} \\ -1 & e^{2x} \end{pmatrix}.$$

Problem 3. Find a particular solution of the differential equation

$$\frac{d}{dx}\vec{y} = A\vec{y} + \vec{b}(x), \quad A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \vec{b}(x) = \begin{pmatrix} e^{2x} \\ 0 \end{pmatrix}$$

[Hint: propose $\vec{y}_p = e^{2x}\vec{c}$]

We propose $\vec{y}_p = e^{2x}\vec{c}$. Then we calculate

$$2e^{2x}\vec{c} = Ae^{2x}\vec{c} + e^{2x}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Dividing by e^{2x}, this becomes

$$2\vec{c} = A\vec{c} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore

$$(A - 2I)\vec{c} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Thus

$$\vec{c} = (A - 2I)^{-1}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -3 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/10 \\ -1/10 \end{pmatrix}.$$

A particular solution is therefore

$$\vec{y} = e^{2x}\begin{pmatrix} 3/10 \\ -1/10 \end{pmatrix}.$$