Math 309 Quiz 5 (Groups)

May 25, 2016

Problem 1. Consider the wave problem

$$u_{tt} = c^2 u_{xx}$$

$$u(0,t) = 0, \quad u(L,t) = 0$$

$$u(x,0) = 0, \quad u_t(x,0) = g(x)$$

where here

$$g(x) = \begin{cases} 4x/L, & 0 \le x < L/4 \\ 1, & L/4 \le x < 3L/4 \\ 4(L-x)/L & 3L/4 \le x \le L \end{cases}$$

(a) Find a solution to the wave equation of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi ct/L) \sin(n\pi x/L)$$

for some constants a_n

(b) If the vibration of a string is described by $u(x,t)$ as in (a)then the energy in frequency $\omega_n = n/(2L)$ is given by

$$E_n = \frac{1}{2} K c^2 n a_n^2$$

where K is some constant having to do with the material properties of the string. Use your answer in (a) to plot the energy E_n as a function of the frequency f_n (take $K = 1, c = 1$). In which frequency is the energy largest? What happens to the energy as $n \to \infty$?

Solution 1.

(a) Recall that the a_n are related to the coefficients in the sine series expansion of $f(x)$, so we calculate that first. To do so, we reflect $g(x)$ oddly and then extend it $2L$-periodically, and take the Fourier transform. Doing so, we have that

$$g(x) = \sum_{n=1} q_n \sin(n\pi x/L)$$

where

$$q_n = \frac{1}{L}\int_{-L}^{L} g(x)\sin(n\pi x/L) = \frac{2}{L}\int_0^L g(x)\sin(n\pi x/L)dx.$$

Then since $g(x)$ is continuous with piecewise continuous first derivative, we may use integration by parts:

$$\int_0^L g(x)\sin(n\pi x/L)dx = \frac{-L}{n\pi}g(x)\cos(n\pi x/L)|_0^L + \frac{L}{n\pi}\int_0^L g'(x)\cos(n\pi x/L)dx = \frac{L}{n\pi}\int_0^L g'(x)\text{co}$$

Then since

$$g'(x) = \begin{cases} 4/L, & 0 \le x < L/4 \\ 0, & L/4 \le x < 3L/4 \\ -4/L & 3L/4 \le x \le L \end{cases}$$

we have that

$$\frac{L}{n\pi}\int_0^L g'(x)\cos(n\pi x/L)dx = \frac{L}{n\pi}\left(\int_0^{L/4}\frac{4}{L}\cos(n\pi x/L)dx + \int_{3L/4}^{L}\frac{-4}{L}\cos(n\pi x/L)dx\right) = \frac{4L}{n^2\pi^2}$$

Therefore we see

$$q_n = \frac{8}{n^2\pi^2}\left(\sin(n\pi/4) + \sin(3n\pi/4)\right).$$

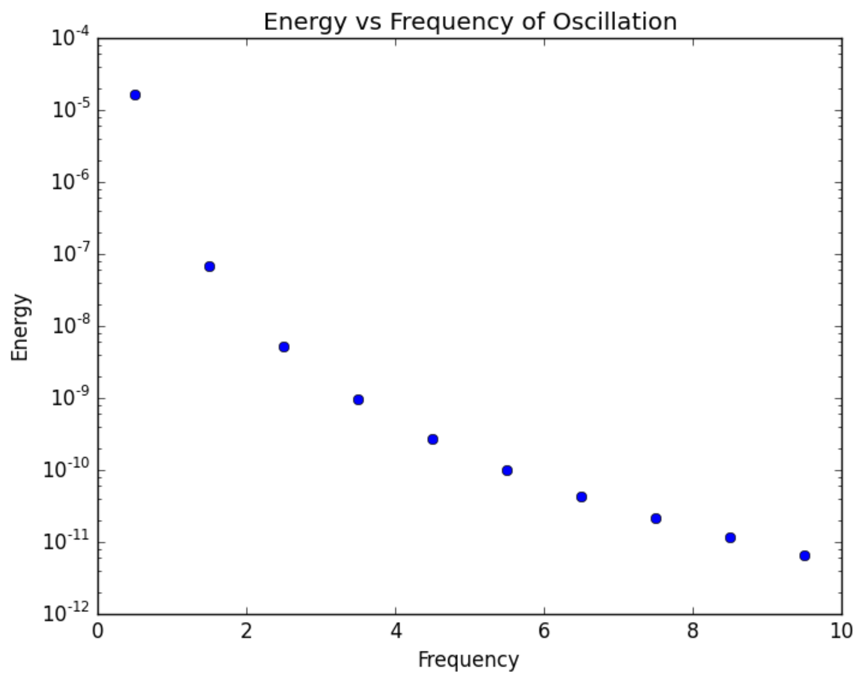
Then the a_n from the statement of the problem are related to the q_n that we just calculated by $a_n = q_n\frac{L}{n\pi c}$. Consequently we find

$$a_n = \frac{8L}{n^3\pi^3 c}\left(\sin(n\pi/4) + \sin(3n\pi/4)\right).$$

(b) The energy in frequency $\omega_n = n/(2L)$ is given by

$$E_n = \frac{1}{2}K\frac{64L^2}{n^5\pi^6}\left(\sin(n\pi/4) + \sin(3n\pi/4)\right)^2$$

Plotting E_n with respect to ω_n, we get the following graph:



In particular, the most energy is in the lowest frequency, and the energy in each frequency dies off rapidly as $\omega_n \to \infty$.

Problem 2. Consider the same wave problem as in Problem 1

(a) Use the method of d'Alembert to find a solution of the wave problem in Problem 1, with $c = 4$ and $L = 1$.

(b) Plot $u(x, 1/8)$.

Solution 2.

(a) To use d'Alembert's method, we extend $g(x)$ oddly and $2L$-periodically. Therefore the integral $G(x)$ of $g(x)$ will be even and 2-periodic, with

$$G(x) = \begin{cases} 2x^2, & 0 \le x < 1/4 \\ x - 1/8, & 1/4 \le x < 3/4 \\ -5/4 + 4x - 2x^2, & 3/4 \le x \le 1 \end{cases}$$

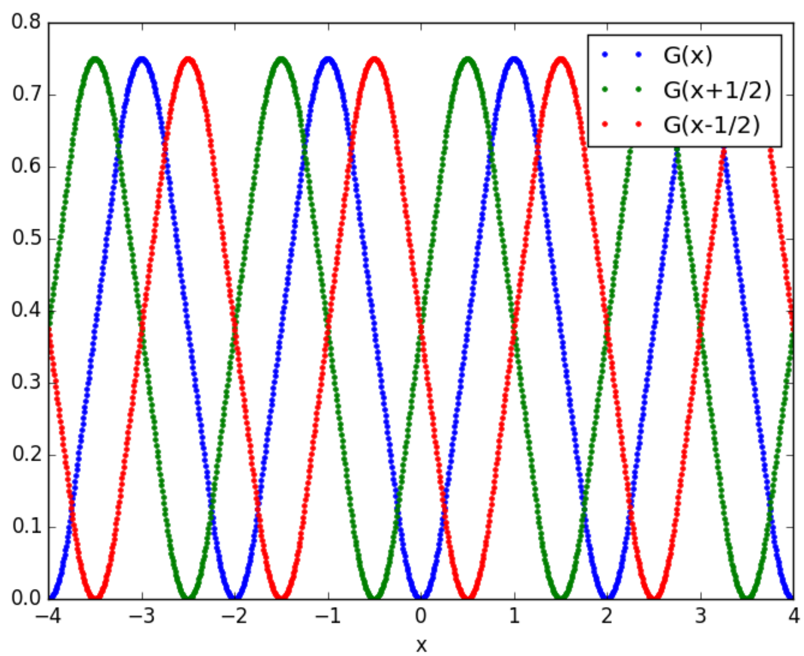
Since $c = 4$, the solution is then:

$$u(x,t) = \frac{1}{2c}(G(x+4t) - G(x-4t)).$$

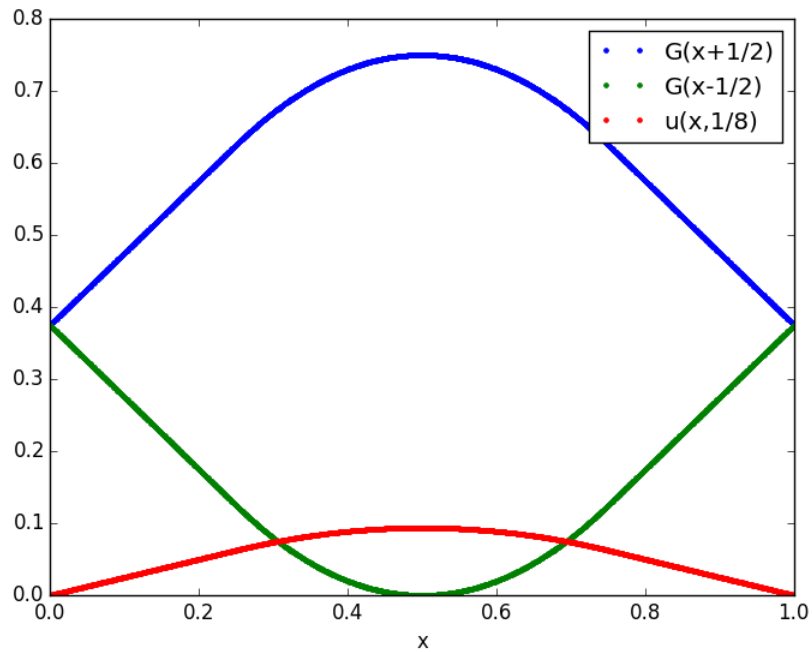
(b) From (a), we have that

$$u(x, 1/8) = \frac{1}{8}(G(x+1/2) - G(x-1/2)),$$

which is an eighth of the difference of $G(x)$ shifted to the left by 1/2 and $G(x)$ shifted to the right by 1/2. A plot of $G(x)$ (which again is even and 2-periodic) and the shifted functions $G(x+1/2)$ and $G(x-1/2)$ is:



Then to get the solution, we take $G(x+1/2)$, subtract $G(x-1/2)$, multiply by 1/8, and restrict once more to the domain $0 \leq x \leq 1$. Below is a graph of $G(x+1/2)$ and $G(x-1/2)$ in the domain $0 \leq x \leq 1$ along with $u(x, 1/8) = \frac{1}{8}(G(x+1/2) - G(x-1/2))$.


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0
x
G(x+1/2)
G(x-1/2)
u(x,1/8)