Math 309 Section I
Spring 2017
Midterm
May 3, 2017
Time Limit: 50 Minutes

Name (Print): ______________________________

Student ID: ______________________________

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This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right.

1. (a) (5 points) Find a solution to the initial value problem

$$y_1'(t) = -12y_1(t) + 6y_2(t)$$
$$y_2'(t) = -35y_1(t) + 17y_2(t)$$

satisfying the initial condition $y_1(0) = 1, y_2(0) = 3$.

(b) (5 points) Find a particular solution of the equation

$$y_1'(t) = y_1(t) + 2y_2(t) - 2e^t$$
$$y_2'(t) = 2y_1(t) + y_2(t)$$

Solution 1.

(a) The characteristic polynomial is $x^2 - 5x + 6$, so the eigenvalues are 2 and 3. A fundamental matrix is therefore

$$\Phi(t) = \frac{e^{2t}}{2-3}(A - 3I) + \frac{e^{3t}}{3-2}(A - 2I) = \begin{pmatrix} 15e^{2t} - 14e^{3t} & 6e^{3t} - 6e^{2t} \\ 35e^{2t} - 35e^{3t} & 15e^{3t} - 14e^{2t} \end{pmatrix}.$$

Since $\Phi(0) = I$, this gives the solution of the initial value problem as

$$\Phi(t)\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}4e^{3t} - 3e^{2t}\\10e^{3t} - 7e^{2t}\end{pmatrix}.$$

In other words

$$y_1 = 4e^{3t} - 3e^{2t} \quad \text{and} \quad y_2 = 10e^{3t} - 7e^{2t}.$$

(b) The characteristic polynomial is $x^2 - 2x - 3$, so the eigenvalues are 3 and -1. Neither of these appears as an exponent in the nonhomogeneous term, so we can use the method of undetermined coefficients to find a particular solution. We propose a solution of the form

$$\vec{y}_p = e^t\vec{c}.$$

Inserting this into the differential equation gives

$$e^t\vec{c} = Ae^t\vec{c} + \begin{pmatrix}-2\\0\end{pmatrix}e^t, \quad \text{where } A = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}.$$

Simplifying, this gives

$$(I - A)\vec{c} = \begin{pmatrix}-2\\0\end{pmatrix},$$

so that

$$\vec{c} = (I - A)^{-1}\begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}0 & -2\\-2 & 0\end{pmatrix}^{-1}\begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}0 & -1/2\\-1/2 & 0\end{pmatrix}^{-1}\begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

Thus the desired particular solution is $\vec{y}_p = \binom{0}{1}e^t$, or equivalently $y_{1p} = 0$ and $y_{2p} = e^t$.

2. (10 points) Consider the family of matrices

$$A = \begin{pmatrix} 1 & 3 \\ 2 & c \end{pmatrix}.$$

Determine for which values of c the equilibrium point at the origin of the differential equation

$$\frac{d}{dt}\vec{y} = A\vec{y}$$

is asymptotically stable, asymptotically unstable, spirally stable, spirally unstable, or a saddle.

Solution 2. The trace of A is $1 + c$, while the determinant of A is $c - 6$. Therefore for this family of matrices, $\det(A) = \text{tr}(A) - 7$ and this means

- when $\text{tr}(A) < 7$, the equilibrium point at the origin is a saddle
- when $\text{tr}(A) > 7$, the equilibrium point at the origin is exponentially unstable

Note that there are no (real) values of c for which A would have complex eigenvalues, so we never have spiral behavior.

3. For each of the following statements, write TRUE if the statement is TRUE, and FALSE if the statement is false. If the statement is false, **also provide a counter-example**.

   (a) (2 points) If A is an $n \times n$ matrix with an eigenvalue λ of algebraic multiplicity m_a and geometric multiplicity m_g, then the largest possible degree of a generalized eigenvector with eigenvalue λ is $m_a - m_g + 1$.

   (b) (2 points) Suppose A is an 3×3 square matrix, and that A has eigenvalues λ_1 and λ_2 which have algebraic multiplicity 2 and 1, respectively. Then A is not diagonalizable.

   (c) (2 points) Suppose A is an $n \times n$ nondegenerate matrix, and that A has 3 as an eigenvalue with algebraic multiplicity n. Then $A = 3I$.

   (d) (2 points) If A is a 2×2 matrix with $A^2 = -I$, then A must have non-real entries (ie. entries with nonzero imaginary components).

   (e) (2 points) If a square matrix A is nondegenerate, then A is also invertible.

Solution 3.

(a) TRUE.

(b) FALSE. For example:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

(c) TRUE.

(d) FALSE. For example

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(e) FALSE. For example the zero matrix is nondegenerate (it's diagonal!) but is not invertible.

4. For each of the following, give an example if an example exists. If an example does not exist, then write DOES NOT EXIST in big bold letters.
   (a) (2 points) A 3×3 matrix with a generalized eigenvector of degree 3.
   (b) (2 points) Two 2×2 matrices A and B with $e^{A+B} \neq e^A e^B$.
   (c) (2 points) Two different solutions of $\frac{d}{dt}\vec{y} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\vec{y}$ which both satisfy the initial condition $\vec{y}(0) = \binom{2}{1}$.
   (d) (2 points) A collection of linearly independent functions whose Wronskian is zero.
   (e) (2 points) A nonzero 2×2 matrix A whose square is the zero matrix.

Solution 4.

(a) One example is the 3×3 Jordan block

$$\begin{pmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{pmatrix}.$$

(b) Note that $e^{A+B} = e^A e^B$ if A and B commute. Therefore we should choose A and B to not commute. For example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then we have

$$e^A = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \quad e^B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad e^{A+B} = \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}$$

In particular $e^A e^B \neq e^{A+B}$.

(c) DOES NOT EXIST

(d) One example is

$$\vec{f}_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad \vec{f}_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(e) One example is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

5. Consider the 3×3 matrix

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

(a) (3 points) Determine the eigenvalues of A and their algebraic multiplicities

(b) (3 points) For each eigenvalue λ of A, determine its geometric multiplicity and the eigenspace $E_\lambda(A)$

(c) (4 points) Find an invertible matrix P and a matrix N in Jordan normal form such that $P^{-1}AP = N$

Solution 5.

(a) The eigenvalues can be read off of the main diagonal. They are 4 and -4, with algebraic multiplicities 2 and 1, respectively.

(b) We calculate

$$E_4(A) = \mathcal{N}(A - 4I) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -8 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

$$E_{-4}(A) = \mathcal{N}(A+4I) = \mathcal{N}\left(\begin{bmatrix} 8 & 1 & 2 \\ 0 & 8 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 0 & 15/64 \\ 0 & 1 & 1/8 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} -15 \\ -8 \\ 64 \end{bmatrix}\right\}.$$

In particular, this says that the geometric multiplicity of 4 and -4 are both 1.

(c) To acquire the matrix P, we must find a generalized eigenvector of degree 2 for the eigenvalue 4. To do so, we calculate the nullspace of $(A - 4I)^2$:

$$\begin{aligned} \mathcal{N}\left((A - 4I)^2\right) &= \mathcal{N}\left(\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -8 \end{bmatrix}^2\right) = \mathcal{N}\left(\begin{bmatrix} 0 & 0 & -15 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{bmatrix}^2\right) \\ &= \mathcal{N}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \end{aligned}$$

Everything in this nullspace is a generalized eigenvector with eigenvalue λ. Moreover since $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is not an eigenvector, it has degree 2. Setting $\vec{v} = (A - 4I)\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} -15 \\ -8 \\ 64 \end{bmatrix}$ we have $P^{-1}AP = N$ for

$$P = [\vec{u}\ \vec{v}\ \vec{w}] = \begin{bmatrix} -15 & 1 & 0 \\ -8 & 0 & 1 \\ 64 & 0 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

BONUS PROBLEMS:

1. Bonus 1: Consider the two 3×3 matrices

$$A = \begin{pmatrix} 17 & 18 & 31 \\ -4 & -2 & -9 \\ -4 & -5 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & -1 & -3 \\ -8 & 5 & 8 \\ 4 & -1 & -1 \end{pmatrix}$$

Find a matrix P such that $P^{-1}AP = B$.

Solution 1. The way to do this is to use the eigenvalues of A and B. Both A and B have only one eigenvalue 3, with algebraic multiplicity 3 and geometric multiplicity 1. Therefore if we take

$$\vec{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v} = (A - 3I)\vec{w} = \begin{pmatrix} 18 \\ -5 \\ -5 \end{pmatrix} \quad \vec{u} = (A - 3I)\vec{v} = \begin{pmatrix} 7 \\ -2 \\ -2 \end{pmatrix}$$

then

$$P_1 = [\vec{u}\ \vec{v}\ \vec{w}] = \begin{pmatrix} 7 & 18 & 0 \\ -2 & -5 & 1 \\ -2 & -5 & 0 \end{pmatrix}$$

satisfies

$$P_1^{-1}AP_1 = J_3(3) = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Similarly, if we take

$$\vec{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v} = (B - 3I)\vec{w} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad \vec{u} = (B - 3I)\vec{v} = \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}$$

then

$$P_2 = [\vec{u}\ \vec{v}\ \vec{w}] = \begin{pmatrix} -1 & -1 & 0 \\ 4 & 2 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$

satisfies

$$P_2^{-1}BP_2 = J_3(3) = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Consequently $P = P_1P_2^{-1}$ satisfies

$$P^{-1}AP = P_2P_1^{-1}AP_1P_2^{-1} = P_2J_3(3)P_2^{-1} = B.$$

Thus the desired matrix is

$$P = P_1P_2^{-1} = \begin{pmatrix} -29 & 0 & 11 \\ 8 & 1 & -1 \\ 8 & 0 & -3 \end{pmatrix}.$$

2. Bonus 2: Consider the Jordan block

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Determine the value of J^{2017}.

Solution 2. To calculate this, we use a trick similar to the one we used to calculate matrix exponentials of degenerate matrices:

$$J^{2017} = (N + \lambda I)^{2017} \quad \text{for} N = J - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and $N^3 = 0I$ so that $N^4 = 0I, N^5 = 0I, ...$ Then by the binomial theorem, we calculate

$$\begin{aligned} (N + \lambda I)^{2017} &= \sum_{n=0}^{2017} \binom{2017}{n} N^n (\lambda I)^{2017-n} \\ &= \binom{2017}{0} (\lambda I)^{2017} + \binom{2017}{1} N (\lambda I)^{2016} + \binom{2017}{2} N^2 (\lambda I)^{2015} \\ &= \lambda^{2017} I + 2017 N \lambda^{2016} I + \frac{(2017)(2016)}{2} N^2 \lambda^{2015} I \\ &= \begin{pmatrix} \lambda^{2017} & 2017\lambda^{2016} & (2017)(2016)\lambda^{2015}/2 \\ 0 & \lambda^{2017} & 2017\lambda^{2016} \\ 0 & 0 & \lambda^{2017} \end{pmatrix} \end{aligned}$$