Matrix Exponential Formulas

Linear Analysis

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Abstract

We present some tricks for quickly calculating the matrix exponential of certain special classes of matrices, such as 2×2 matrices.

1 2×2 matrix exponential formulas

In this section, we address how to quickly calculate a matrix exponential by hand for a 2×2 matrix A. The answer, it turns out, will depend on the particulars of the eigenvalues of A. There are three possible cases

(a) the eigenvalues of A are real and distinct

(b) the eigenvalues of A are real and the same

(c) the eigenvalues of A are a complex conjugate pair

We already know how to calculate $\exp(At)$ in any of these cases – we conjugate A to get a matrix in Jordan normal form, we take the exponential of that matrix, and we conjugate back. This process can feel a bit tedious, since it involves finding the eigenspaces for each eigenvalue, possibly finding generalized eigenvectors, calculating products of matrices, etc. Moreover, it can sometimes feel that to calculate several matrix exponentials by hand without creating some sort of algebraic catastrophe along the way will require a small miracle. Thus we are motivated to determine simple explicit equations for the value of A in each of the above cases. We do so below, and our results are summarized in the following table Using the above table, if we figure

Eigenvalues of A	$\exp(At)$
r_1, r_2 real and distinct	$e^{r_1 t}\frac{1}{r_1 - r_2}(A - r_2 I) - e^{r_2 t}\frac{1}{r_1 - r_2}(A - r_1 I)$
r repeated twice	$e^{rt}I + e^{rt}t(A - rI)$
$a \pm ib$ complex conjugate pair	$e^{at}\cos(bt)I + \frac{1}{b}e^{at}\sin(bt)(A - aI)$

out the eigenvalues of A, then we can insert them into the relevant formula above to obtain the desired matrix exponential, without the bother of diagonalization, Jordan normal form, etc.

Example 1. Consider the matrix

$$A = \begin{pmatrix} -27 & 10 \\ -75 & 28 \end{pmatrix}$$

We calculate the characteristic polynomial of A to be $p_A(x) = x^2 - x - 6 = (x-3)(x+2)$. Therefore the eigenvalues are real and distinct, given by $r_1 = 3, r_2 = -2$. We then calculate

$$\begin{aligned}
\exp(At) &= e^{r_1 t}\frac{1}{r_1 - r_2}(A - r_2 I) - e^{r_2 t}\frac{1}{r_1 - r_2}(A - r_1 I) \\
&= e^{3t}\frac{1}{5}(A + 2I) - e^{-2t}\frac{1}{5}(A - 3I) \\
&= e^{3t}\frac{1}{5}\begin{pmatrix} 29 & 10 \\ -75 & 30 \end{pmatrix} - e^{-2t}\frac{1}{5}\begin{pmatrix} 24 & 10 \\ -75 & 25 \end{pmatrix} = \begin{pmatrix} (29/5)e^{3t} - (24/5)e^{-2t} & 2e^{3t} - 2e^{2t} \\ -15e^{3t} + 15e^{-2t} & 6e^{3t} - 5e^{-2t} \end{pmatrix}
\end{aligned}$$

Example 2. Consider the matrix

$$A = \begin{pmatrix} 6 & -1 \\ 9 & 0 \end{pmatrix}$$

We calculate the characteristic polynomial of A to be $p_A(x) = x^2 - 6x + 9 = (x - 3)(x - 3)$. Therefore the eigenvalue is $r = 3$, repeated twice. We then calculate

$$\begin{aligned}
\exp(At) &= e^{rt} I + e^{rt}(A - rI) \\
&= e^{3t} I + e^{3t}(A - 3I) \\
&= e^{3t} I + e^{3t} t \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} = \begin{pmatrix} e^{3t} + 3te^{3t} & -te^{3t} \\ 9te^{3t} & e^{3t} - 3te^{3t} \end{pmatrix}
\end{aligned}$$

Example 3. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 \\ -7 & 1 \end{pmatrix}$$

We calculate the characteristic polynomial of A to be $p_A(x) = x^2 - 2x + 29$. From the quadratic formula, we find that the roots are $1 \pm 2\sqrt{7}i$. This is the complex conjugate pair case $a \pm ib$ with $a = 1$ and $b = 2\sqrt{7}$. We then calculate

$$\begin{aligned}
\exp(At) &= e^{at}\cos(bt) I + \frac{1}{b} e^{at}\sin(bt)(A - rI) \\
&= e^{t}\cos(2\sqrt{7}t) I + \frac{1}{2\sqrt{7}} e^{t}\sin(2\sqrt{7}t)(A - I) \\
&= e^{t}\cos(2\sqrt{7}t) I + \frac{1}{2\sqrt{7}} e^{t}\sin(2\sqrt{7}t)\begin{pmatrix} 0 & 4 \\ -7 & 0 \end{pmatrix} = \begin{pmatrix} e^{t}\cos(2\sqrt{7}t) & \frac{2}{\sqrt{7}} e^{t}\sin(2\sqrt{7}t) \\ \frac{-7}{2\sqrt{7}} e^{t}\sin(2\sqrt{7}t) & e^{t}\cos(2\sqrt{7}t) \end{pmatrix}
\end{aligned}$$

2 3×3 matrix exponential formulas

In this section, we address how to quickly calculate a matrix exponential by hand for a 3×3 matrix A. The answer again depends on the eigenvalues of A and their multiplicities. We consider three possibilities.

(a) A has one eigenvalue λ of algebraic multiplicity 3

(b) A has two eigenvalues λ, μ of algebraic multiplicity 1 and 2, respectively

(c) A has three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$

Again, we can calculate $\exp(At)$ by finding the Jordan normal form of A. However, this can be a difficult and algebraically lengthy mathematical journey. Instead, we can derive very fast formulas for these which do not rely on diagonalization. The following three propositions give equations for $3x3$ matrix exponentials

Proposition 1. *Let A be a 3×3 matrix with a single eigenvalue λ of algebraic multiplicity 3. Then*

$$\exp(At) = (I + (A - \lambda I)t + \frac{1}{2}(A - \lambda I)^2 t^2)e^{\lambda t}$$

Proposition 2. *Let A be a 3×3 matrix with two different eigenvalues λ, μ of algebraic multiplicities 1 and 2, respectively. Then*

$$\exp(At) = \frac{e^{\mu t}}{\mu - \lambda}(I + (A - \mu I)t)(A - \lambda I) + \frac{e^{\lambda t}}{(\lambda - \mu)^2}(A - \mu I)^2.$$

Proposition 3. *Let A be a 3×3 matrix with three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Then*

$$\begin{aligned}\exp(At) = &\frac{e^{\lambda_1 t}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}(A - \lambda_2 I)(A - \lambda_3 I) \\ &+ \frac{e^{\lambda_2 t}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}(A - \lambda_1 I)(A - \lambda_3 I) \\ &+ \frac{e^{\lambda_3 t}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}(A - \lambda_1 I)(A - \lambda_2 I)\end{aligned}$$

Using the above three theorems, if we figure out what the eigenvalues of A, along with their algebraic multiplicities, then we can immediately calculate the matrix exponential. As in the 2×2 case, this avoids the need to find a cyclic basis, use Jordan normal form, etc. We give several examples of these propositions in action below. In all of our examples, the matrices are upper triangular for ease of computation. However the theorems work, of course, for any matrices.

Example 4. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

This has exactly one eigenvalue $\lambda = 2$ with algebraic multiplicity 2. Therefore by Proposition 1, we have

$$\exp(At) = (I + (A - 2I)t + \frac{1}{2}(A - 2I)^2 t^2)e^{2t} = \begin{pmatrix} 1 & t & t + 3t^2/2 \\ 0 & 1 & 3t \\ 0 & 0 & 1 \end{pmatrix} e^{2t}.$$

Example 5. Consider the matrix

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

This has eigenvalues -1 with algebraic multiplicity 2 and 2 with algebraic multiplicity 1. Therefore we have

$$\begin{aligned} \exp(At) &= \frac{e^{-t}}{-3}(I + (A+I)t)(A - 2I) + \frac{e^{2t}}{9}(A+I)^2 \\ &= \begin{pmatrix} 1 & -2/3 & -1 + 7t/3 \\ 0 & 0 & -1/3 \\ 0 & 0 & 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 & 2/3 & 2/9 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix} e^{2t} \end{aligned}$$

Example 6. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

This has eigenvalues $1, 4, 6$ all of which have algebraic multiplicity 1. Therefore we calculate

$$\begin{aligned} \exp(At) &= \frac{e^{t}}{15}(A - 4I)(A - 6I) + \frac{e^{4t}}{-6}(A - I)(A - 6I) + \frac{e^{6t}}{10}(A - I)(A - 4I) \\ &= \begin{pmatrix} 2/3 & -2/3 & 1/15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{t} + \begin{pmatrix} 0 & 2/3 & -3/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{pmatrix} e^{4t} + \begin{pmatrix} 0 & 0 & 8/5 \\ 0 & 0 & 5/2 \\ 0 & 0 & 1 \end{pmatrix} e^{6t}. \end{aligned}$$

3 $n \times n$ matrix exponential formulas

What about matrix exponential formulas for matrices larger than 3×3? There are tricks for these, too. One trick, if a matrix A is diagonalizable, is known as **Sylvester's Formula**. It's application is presented in the following theorem.

Theorem 1. *Suppose that A is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\exp(At) = \sum_{i=1}^{n} e^{\lambda_i t} \prod_{j \neq i} \frac{1}{\lambda_i - \lambda_j}(A - \lambda_j I).$$

Thus for diagonalizable matrices, we may again immediately write down the matrix exponential in terms of the original matrix without fooling around with diagonalization. However, for this formula to work, it's very important that the matrix A be diagonalizable. For nondiagonalizable A, the situation is a bit more complicated. We also have a simple formula in the case that A has exactly one eigenvalue repeated n times.

Theorem 2. *Suppose that A in an $n \times n$ matrix with exactly one eigenvalue λ with algebraic multiplicity n. Then*

$$\exp(At) = e^{\lambda t} \sum_{j=1}^{n} \frac{1}{j!}(A - \lambda I)^j t^j.$$

Fooling around with shortcuts like this can be a lot of fun. How about you try to use it to calculate the matrix exponential of some interesting matrices?