MATH 309: Homework #1

Due on: April 10, 2017

Problem 1 *Matrix Algebra*

Let A and B be the matrices

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Determine values of the following

- AB
- BA
- $3A + 2B$

. . . .

Problem 2 *Matrix Puzzles*

(a) Find a nonzero square matrix A with $A^2 = 0$ (here 0 means the zero matrix)

(b) Find a square matrix J with real entries satisfying $J^2 = -I$

(c) Find an invertible matrix P with $P^{-1} = P^\dagger$ (here $P^\dagger$ refers to the conjugate transpose of P)

(d) Find a nonzero square matrix P with $P^2 = P$

(e) Find all possible 2×2 matrices X satisfying the equation $X^2 - 2X + I = 0$

. . . .

Problem 3 *Matrix Inverses*

For each of the following matrices, find the inverse of the matrix or explain why it doesn't have one

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 7 \\ 3 & 4 & 1 \\ 7 & 10 & 9 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 3 & 0 \\ 3 & 2 & 1 \\ 5 & 2 & 5 \end{pmatrix}$

. . . .

Problem 4 *Linear Systems*

For each of the following linear systems of equations, either find the general solution, or show that no solution exists.

(a)

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 - 2x_3 &= 2 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ 3x_1 + 2x_2 + x_3 &= 1 \\ 5x_1 + 2x_2 + 5x_3 &= 1 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + 2x_2 + 7x_3 &= 1 \\ 3x_1 + 4x_2 + x_3 &= 0 \\ 7x_1 + 10x_2 + 9x_3 &= 1 \end{aligned}$$

(d)

$$\begin{aligned} x_1 + 2x_2 + 7x_3 &= 0 \\ 3x_1 + 4x_2 + x_3 &= 0 \\ 7x_1 + 10x_2 + 9x_3 &= 0 \end{aligned}$$

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Problem 5 *Eigenvectors and Eigenvalues*

For each of the following matrices, determine the following information

(i) the eigenvalues

(ii) the algebraic and geometric multiplicity of each eigenvalue

(iii) a basis for the eigenspace of each eigenvalue

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

. . . .

Problem 6 *Eigenvectors and Linear Independence*

Suppose that A is an $n \times n$ matrix and that $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with eigenvalues λ_1 and λ_2, respectively. Show that if $\lambda_1 \neq \lambda_2$, then $\vec{v}_1$ and $\vec{v}_2$ must be linearly independent. (Here, by "show", we mean make a formal argument using both math and complete sentences.)

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Problem 7 *First Order Homogeneous Linear Systems of Ordinary Differential Equations with Constant Coefficients*

Find the general solution of each of the following systems of first order homogeneous linear ordinary differential equations with constant coefficients

(a)

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_1 + 2x_2 \end{aligned}$$

(b)

$$x_1' = 3x_1 + x_2$$
$$x_2' = 2x_1 + 2x_2$$

(c)

$$x_1' = x_1 + x_2$$
$$x_2' = x_2$$

. . . .

Problem 8 *Solution Space*

Let $A(x) = (a_{ij}(x))$ be an $n \times n$ matrix, with the functions $a_{ij}(x)$ continuous on the interval (α, β) for all i, j. Consider the differential equation

$$\frac{d}{dx}\vec{y}(x) = A(x)\vec{y}(x).$$

(a) Explain why the set of solutions to this equation on the interval (α, β) is a vector space

(b) Explain why the dimension of the solution space on the interval (α, β) is n-dimensional. One way to do so is as follows. Let t_0 be a fixed number in (α, β) and for all $1 \leq i \leq n$, let $\vec{e}_i$ represent the vector whose entries are all 0, except the i'th which is 1. Then use the next few arguments.

 (i) Explain why there exists a unique solution $\vec{y}_i(x)$ to the differential equation satisfying $\vec{y}_i(t_0) = \vec{e}_i$ for all $1 \leq i \leq n$. (Hint: Existence/uniqueness theorem)

 (ii) Suppose $\vec{y}$ is a solution, with $\vec{y}(t_0) = \vec{c}$, and let $c_1, c_2, \ldots, c_n$ be the entries in the vector $\vec{c}$. Explain why

 $$\vec{y}(t) = c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + \cdots + c_n\vec{y}_n(t).$$

 Since $\vec{y}$ was an arbitrary solution, this shows that the set $\{\vec{y}_1, \ldots, \vec{y}_n\}$ spans the space of solutions, as we can write any solution as a linear combination of $y_1, \ldots, y_n$.

 (iii) Suppose that

 $$c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + \cdots + c_n\vec{y}_n(t) = \vec{0},$$

 for some constants $c_1, \ldots, c_n$. Evaluating at $t = t_0$, this says

 $$c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_n\vec{e}_n = \vec{0}.$$

 Explain why this implies $c_1 = 0, c_2 = 0, \ldots, c_n = 0$, and then explain why this implies the set $\{\vec{y}_1, \ldots, \vec{y}_n\}$ is a set of linearly independent vectors.

(iv) Using the results of (ii) and (iii), we see that $\{\vec{y}_1, \ldots, \vec{y}_n\}$ is a basis for the space of solutions of the differential equation. Explain why this means that the space of solutions is n-dimensional.

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Problem 9 *Wronskian Issues*

Let $A(x) = (a_{ij}(x))$ be an $n \times n$ matrix, with the functions $a_{ij}(x)$ continuous on the interval (α, β) for all i, j. Consider the differential equation

$$\frac{d}{dx}\vec{y}(x) = A(x)\vec{y}(x).$$

Recall that the Wronskian $W[\vec{y}_1(x), \ldots, \vec{y}_n(x)]$ of solutions $\vec{y}_1(x), \ldots, \vec{y}_n(x)$ is nonzero on (α, β) if and only if the solutions are linearly independent. It's *very important* here that the functions we are considering are solutions to the differential equation $\vec{y}'(x) = A\vec{y}(x)$. To demonstrate this, consider the following functions

$$\vec{y}_1(x) = \begin{pmatrix} 1 \\ x \end{pmatrix}, \quad \vec{y}_2(x) = \begin{pmatrix} e^x \\ xe^x \end{pmatrix}$$

(a) Show that $W[\vec{y}_1(x), \vec{y}_2(x)]$ is identically 0

(b) Despite this, show that $\vec{y}_1(x)$ and $\vec{y}_2(x)$ are actually linearly independent

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