MATH 309: Homework #1

Due on: April 10, 2016

Problem 1 *Matrix Algebra*

Let A and B be the matrices

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Determine values of the following

- AB
- BA
- $3A + 2B$

. . . .

Solution 1.

- $$AB = \begin{pmatrix} 4 & 5 & 6 \\ 10 & 11 & 12 \\ -2 & -1 & 0 \end{pmatrix}$$
- $$BA = \begin{pmatrix} 2 & 4 & 0 \\ 5 & 7 & 3 \\ 8 & 10 & 6 \end{pmatrix}$$
- $$3A + 2B = \begin{pmatrix} 5 & 1 & 9 \\ 5 & 13 & 15 \\ 17 & 19 & 15 \end{pmatrix}$$

Problem 2 *Matrix Puzzles*

(a) Find a nonzero square matrix A with $A^2 = 0$ (here 0 means the zero matrix)

(b) Find a square matrix J with real entries satisfying $J^2 = -I$

(c) Find an invertible matrix P with $P^{-1} = P^\dagger$ (here $P^\dagger$ refers to the conjugate transpose of P)

(d) Find a nonzero square matrix P with $P^2 = P$

(e) Find all possible 2×2 matrices X satisfying the equation $X^2 - 2X + I = 0$

. . . .

Solution 2.

(a) This will be satisfied if A is a 2×2 matrix all of whose eigenvalues are zero, eg.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(b) We are looking for a matrix satisfying $J^2 = -I$. This means that the eigenvalues of J need to be square roots of -1. We know from class that we can get a 2×2 real matrix J with eigenvalues like this by letting J be the matrix corresponding to a rotation by a right angle. One such rotation is $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(c) Matrices satisfying the property that $P^{-1} = P^\dagger$ are called **unitary** and correspond to choices of orthonormal bases in $\mathbb{R}^n$. In other words, we can form such a 2×2 matrix by choosing *any* orthonormal basis for $\mathbb{R}^2$ and using the basis vectors as column vectors for the matrix. The standard basis $\binom{1}{0}, \binom{0}{1}$ is one example of an orthonormal basis, and it corresponds to the identity matrix. Another orthonormal basis is $\binom{1/\sqrt{2}}{1/\sqrt{2}}, \binom{1/\sqrt{2}}{-1/\sqrt{2}}$ and corresponds to the unitary matrix $P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

(d) Matrices satisfying this property are called **projection matrices**. A simple example is again $P = I$.

(e) This seems to be a bit trickier! There are brute-force methods of figuring this out, simpler methods, and many things in between. One thing that it's always helpful to ask when considering a question like this is what do the eigenvalues of X have to be? If λ is an eigenvalue of X with eigenvector $\vec{v}$, then

$$0\vec{v} = (X^2 - 2X + I)\vec{v} = X^2\vec{v} - 2X\vec{v} + \vec{v} = \lambda^2\vec{v} - 2\lambda\vec{v} + \vec{v} = (\lambda^2 - 2\lambda + 1)\vec{v}.$$

Therefore $\lambda^2 - 2\lambda + 1 = 0$, and this means that $\lambda = 1$. Therefore the only eigenvalues X can have are 1 and 1. In this way, the real question we are asking is what 2×2 matrices have 1 (with algebraic multiplicity 2) as their only eigenvalue?

This is an excellent question! Remember, these are determined by the characteristic polynomial. Note that if A is the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$p_A(x) = x^2 - (a+d)x + ad - bc.$$

The roots of this polynomial are

$$\frac{1}{2}(a+d) \pm \frac{1}{2}\sqrt{(a+d)^2 - 4(ad-bc)}.$$

Therefore for all the eigenvalues to be 1, we will need $a + d = 2$ and $ad - bc = 1$. Thus the answer is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 2,\ ad - bc = 1 \right\}.$$

Problem 3 *Matrix Inverses*

For each of the following matrices, find the inverse of the matrix or explain why it doesn't have one

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 7 \\ 3 & 4 & 1 \\ 7 & 10 & 9 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 3 & 0 \\ 3 & 2 & 1 \\ 5 & 2 & 5 \end{pmatrix}$

. . . .

Solution 3.

(a) doesn't have one (wrong shape)

(b) the inverse is $\begin{pmatrix} -1/10 & 2/5 \\ -3/5 & 1/5 \end{pmatrix}$

(c) doesn't have one (determinant is zero)

(d) the inverse is $\begin{pmatrix} -8/22 & 15/22 & -3/22 \\ 10/22 & -5/22 & 1/22 \\ 4/22 & -13/22 & 7/22 \end{pmatrix}$

Problem 4 *Linear Systems*

For each of the following linear systems of equations, either find the general solution, or show that no solution exists.

(a)

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 - 2x_3 &= 2 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ 3x_1 + 2x_2 + x_3 &= 1 \\ 5x_1 + 2x_2 + 5x_3 &= 1 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + 2x_2 + 7x_3 &= 1 \\ 3x_1 + 4x_2 + x_3 &= 0 \\ 7x_1 + 10x_2 + 9x_3 &= 1 \end{aligned}$$

(d)

$$\begin{aligned} x_1 + 2x_2 + 7x_3 &= 0 \\ 3x_1 + 4x_2 + x_3 &= 0 \\ 7x_1 + 10x_2 + 9x_3 &= 0 \end{aligned}$$

. . . .

Solution 4.

(a) We form the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & -2 & 2 \end{array}\right)$$

The RREF is

$$\left(\begin{array}{ccc|c} 1 & 0 & -7 & 5 \\ 0 & 1 & 5 & -3 \end{array}\right)$$

and this gives us the general solution

$$x = 5 + 7t, \quad y = -3 - 5t, \quad z = t$$

(b) We form the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & 2 & 5 & 1 \end{array}\right)$$

The RREF is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 12/22 \\ 0 & 1 & 0 & -4/22 \\ 0 & 0 & 0 & -6/22 \end{array}\right)$$

and this gives us the general solution

$$x = 12/22, \quad y = -4/22, \quad z = -6/22$$

(c) We form the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 7 & 1 \\ 3 & 4 & 1 & 0 \\ 7 & 10 & 9 & 1 \end{array}\right)$$

The RREF is

$$\left(\begin{array}{ccc|c} 1 & 0 & -13 & -2 \\ 0 & 1 & 10 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and this gives us the general solution

$$x = -2 + 13t, \quad y = 3/2 - 10t, \quad z = t.$$

(d) We form the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 3 & 4 & 1 & 0 \\ 7 & 10 & 9 & 0 \end{array}\right)$$

The RREF is

$$\left(\begin{array}{ccc|c} 1 & 0 & -13 & 0 \\ 0 & 1 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and this gives us the general solution

$$x = 13t, \quad y = -10t, \quad z = t.$$

Problem 5 *Eigenvectors and Eigenvalues*

For each of the following matrices, determine the following information

(i) the eigenvalues

(ii) the algebraic and geometric multiplicity of each eigenvalue

(iii) a basis for the eigenspace of each eigenvalue

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

. . . .

Solution 5.

(a) The characteristic polynomial of A is $p_A(x) = x^2 - 3x + 1$. The eigenvalues of A are $(3/2) \pm \sqrt{5}/2$. Thus we know right away that the eigenvalues of A both have algebraic and geometric multiplicity 1. Furthermore

$$E_{3/2+\sqrt{5}/2}(A) = \text{span}\left\{\begin{pmatrix} -1/2 + \sqrt{5}/2 \\ 1 \end{pmatrix}\right\},\ E_{3/2-\sqrt{5}/2} = \text{span}\left\{\begin{pmatrix} -1/2 - \sqrt{5}/2 \\ 1 \end{pmatrix}\right\}$$

(b) The characteristic polynomial of A is $p_A(x) = x^2 + x - 1$. The eigenvalues of A are $-1/2 \pm \sqrt{5}/2$. Thus again we are immediately aware that the eigenvalues of A have both algebraic and geometric multiplicity 1. Furthermore

$$E_{-1/2+\sqrt{5}/2}(A) = \text{span}\left\{\begin{pmatrix} -1/2 + \sqrt{5}/2 \\ 1 \end{pmatrix}\right\},\ E_{-1/2-\sqrt{5}/2} = \text{span}\left\{\begin{pmatrix} -1/2 - \sqrt{5}/2 \\ 1 \end{pmatrix}\right\}$$

(c) The characteristic polynomial of A is $p_A(x) = x^2$. There is only one eigenvalue, namely 0, with algebraic multiplicity 2. We calculate the eigenspace of 0 to be

$$E_0(A) = N(A - 0I) = \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}.$$

The geometric multiplicity of 0 is therefore 1.

(d) The characteristic polynomial of A is $p_A(x) = (1-x)(x^2 - 2x + 5)$. Therefore the eigenvalues of A are 1 and $1 \pm 2i$. It follows that the algebraic and geometric multiplicities of all the roots are 1. The eigenspaces are given by

$$E_1(A) = \operatorname{span}\left\{\begin{pmatrix}1\\-3/2\\1\end{pmatrix}\right\},\ E_{1+2i}(A) = \operatorname{span}\left\{\begin{pmatrix}0\\1\\-i\end{pmatrix}\right\},\ E_{1-2i}(A) = \operatorname{span}\left\{\begin{pmatrix}0\\1\\i\end{pmatrix}\right\}$$

(e) The characteristic polynomial of A is $p_A(x) = -(x-1)(x-2)(x-3)$. Therefore the eigenvalues of A are $1, 2, 3$, and the algebraic and geometric multiplicities of these must all be 1. The corresponding eigenspaces are

$$E_1(A) = \operatorname{span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\},\ E_2(A) = \operatorname{span}\left\{\begin{pmatrix}2\\-1\\0\end{pmatrix}\right\},\ E_3(A) = \operatorname{span}\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$$

Problem 6 *Eigenvectors and Linear Independence*

Suppose that A is an $n \times n$ matrix and that $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with eigenvalues λ_1 and λ_2, respectively. Show that if $\lambda_1 \neq \lambda_2$, then $\vec{v}_1$ and $\vec{v}_2$ must be linearly independent. (Here, by "show", we mean make a formal argument using both math and complete sentences.)

. . . .

Solution 6. This was a problem that a lot of people seemed to struggle with, perhaps because of the difficulty of dealing with abstraction. That's ok – it just takes a little practice. If you have trouble with this sort of question in the future, please don't hesitate to talk with your peers or myself. It's especially important to know if you "understand the problem". If you don't understand what the difficulty that needs to be overcome is, then it is hard to correctly overcome it.

Suppose that A is an $n \times n$ matrix, and that $\vec{v}_1, \vec{v}_2$ are eigenvectors of A with eigenvalues λ_1, λ_2, respectively. Suppose also that $\lambda_1 \neq \lambda_2$. We wish to show that $\vec{v}_1$ and $\vec{v}_2$ must be linearly independent. TO do so, we must show that if c_1, c_2 are constants satisfying

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0},$$

then $c_1 = 0$ and $c_2 = 0$. To do this, we suppose that c_1, c_2 are constants satisfying

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}.$$

Multiplying both sides by the matrix $A - \lambda_1 I$, we then find

$$c_1(A - \lambda_1 I)\vec{v}_1 + c_2(A - \lambda_1 I)\vec{v}_2 = \vec{0}.$$

It is easy to check that $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$ and $(A - \lambda_1 I)\vec{v}_2 = (\lambda_2 - \lambda_1)\vec{v}_2$. Therefore

$$c_2(\lambda_2 - \lambda_1)\vec{v}_2 = \vec{0}.$$

Since $\lambda_2 - \lambda_1 \neq 0$, this implies $c_2 = 0$. Similarly multiplying both sides of the original linear combination

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0},$$

by the expression $(A - \lambda_2 I)$ yeilds $c_1 = 0$. Therefore both c_1 and c_2 must be 0, and we conclude that $\vec{v}_1$ and $\vec{v}_2$ must be linearly independent.

Problem 7 *First Order Homogeneous Linear Systems of Ordinary Differential Equations with Constant Coefficients*

Find the general solution of each of the following systems of first order homogeneous linear ordinary differential equations with constant coefficients

(a)

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_1 + 2x_2 \end{aligned}$$

(b)

$$\begin{aligned} x_1' &= 3x_1 + x_2 \\ x_2' &= 2x_1 + 2x_2 \end{aligned}$$

(c)

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 \end{aligned}$$

. . . .

Solution 7.

(a) We rewrite this in vector form as

$$\vec{y}'(t) = A\vec{y}(t), A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The eigenvectors and eigenvalues of the matrix A were already found in the previous problem. From this we get the general solution

$$E_{-3/2+\sqrt{5}/2}(A) = \text{span}\left\{\begin{pmatrix} -1/2+\sqrt{5}/2 \\ 1 \end{pmatrix}\right\}, \; E_{-3/2-\sqrt{5}/2} = \text{span}\left\{\begin{pmatrix} -1/2-\sqrt{5}/2 \\ 1 \end{pmatrix}\right\}$$

$$\vec{y}(t) = c_1 \begin{pmatrix} -1/2+\sqrt{5}/2 \\ 1 \end{pmatrix} e^{(-3/2+\sqrt{5}2)t} + c_2 \begin{pmatrix} -1/2-\sqrt{5}/2 \\ 1 \end{pmatrix} e^{(-3/2-\sqrt{5}/2)t}.$$

(b) We rewrite this in vector form as

$$\vec{y}'(t) = A\vec{y}(t), A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}.$$

We calculate the eigenvalues of A to be 1 and 4 with corresponding eigenspaces

$$E_1(A) = \text{span}\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\}, \quad E_4(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

Therefore we can automatically write down the general solution as

$$\vec{y}(t) = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

(c) We rewrite this in vector form as

$$\vec{y}'(t) = A\vec{y}(t), A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The matrix A is not diagonalizable, so unfortunately we cannot use the previous method to write down the general solution. However, if we stare at the equations for a second the first thing that we notice is that $x_2' = x_2$. This is an regular old differential equation whose set of solutions is $x_2 = ce^t$ for some arbitrary constant c. Now if we fill this into the first equation, we find

$$x_1' = x_1 + ce^t$$

This again is a regular old linear ordinary differential equation, whose solution we can find using an integrating factor, variation of parameters, etc. The solution is $x_1 = be^t + cte^t$ for some arbitrary constant b. In this way, we've found the general solution! In terms of vectors:

$$\vec{y}(t) = b\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c\begin{pmatrix} t \\ 1 \end{pmatrix} e^t$$

Problem 8 *Solution Space*

Let $A(t) = (a_{ij}(t))$ be an $n \times n$ matrix, with the functions $a_{ij}(t)$ continuous on the interval (α, β) for all i, j. Consider the differential equation

$$\vec{y}'(t) = A(t)\vec{y}(t).$$

(a) Explain why the set of solutions to this equation on the interval (α, β) is a vector space

(b) Explain why the dimension of the solution space on the interval (α, β) is n-dimensional (I am asking you to reproduce the argument we did in lecture)

. . . .

Solution 8.

1. The set of solutions is a vector space because of the **superposition principle**: any linear combination of solutions will also be a solution. Due to this, the set of solutions is closed under addition and scalar multiplication, and therefore a vector space.

2. We repeat the argument provided in class. The main idea is that solutions to the equation on the interval (α, β) are determined by their value at a particular point $t_0 \in (\alpha, \beta)$. Let $\vec{e}_1, \ldots, \vec{e}_n$ be the standard basis for $\mathbb{R}^n$. For $i = 1, 2, \ldots, n$, let $\vec{y}_i(t)$ be a solution to the initial value problem

$$\vec{y}'(t) = A(t)\vec{y}(t), \quad y(t_0) = \vec{e}_i.$$

Such a solution exists by the Existence and Uniqueness Theorem. We claim that $\{\vec{y}_1(t), \vec{y}_2(t), \ldots, \vec{y}_n(t)\}$ is a basis for the space of solutions. Note that if this is true, then it proves that the dimension of the solution space is n, since that is the number of elements in this (hence any) basis. To prove our claim, we must show that $\{\vec{y}_1(t), \vec{y}_2(t), \ldots, \vec{y}_n(t)\}$ spans the space of solutions and is a linearly independent set of vectors.

LINEAR INDEPENDENCE:

We will first prove the linear independence of the set $\{\vec{y}_1(t), \vec{y}_2(t), \ldots, \vec{y}_n(t)\}$. To do this, suppose that $c_1, c_2, \ldots, c_n$ are constants satisfying

$$c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + \cdots + c_n\vec{y}_n(t) = \vec{0}.$$

Then if we evaluate at $t = t_0$, this says

$$c_1\vec{y}_1(t_0) + c_2\vec{y}_2(t_0) + \cdots + c_n\vec{y}_n(t_0) = \vec{0}.$$

Then since $\vec{y}_i(t_0) = \vec{e}_i$, we have that

$$c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_n\vec{e}_n = \vec{0}.$$

However, since $\vec{e}_1, \dots, \vec{e}_n$ is a basis for $\mathbb{R}^n$, this implies that $c_1 = 0, c_2 = 0, \dots, c_n = 0$. Thus $\vec{y}_1(t), \dots, \vec{y}_n(t)$ are linearly independent.

SPAN:

To complete our solution, we need to show that the set of vectors $\{\vec{y}_1(t), \vec{y}_2(t), \dots, \vec{y}_n(t)\}$ spans the space of solutions. With this in mind, suppose that $\vec{z}(t)$ is any solution to the differential equation $\vec{y}'(t) = A(t)\vec{y}(t)$ on the interval (α, β). Let $\vec{v} \in \mathbb{R}^n$ be the value of $\vec{z}(t)$ at $t = t_0$, ie. $\vec{v} := z(\vec{t_0})$. Then since $\vec{e}_1, \dots, \vec{e}_n$ form a basis for $\mathbb{R}^n$, there exist constants $c_1, c_2, \dots, c_n$ such that

$$\vec{v} = c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_n\vec{e}_n.$$

Now consider the function $\vec{w}(t) := c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + \cdots + c_n\vec{y}_n(t)$. By the fact that the space of solutions is a vector space (or equivalently, by the superposition principle), we konw that $\vec{w}(t)$ is also a solution to $\vec{y}'(t) = A(t)\vec{y}(t)$. Moreover the value of $\vec{w}(t)$ at $t = t_0$ is

$$\vec{w}(t_0) = c_1\vec{y}_1(t_0) + c_2\vec{y}_2(t_0) + \cdots + c_n\vec{y}_n(t_0) = c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_n\vec{e}_n = \vec{v}.$$

This means that $\vec{z}(t)$ and $\vec{w}(t)$ are both solutions of the initial value problem

$$\vec{y}'(t) = A(t)\vec{y}(t), \quad y(t_0) = \vec{v}.$$

The Existence and Uniqueness Theorem tells us that there is only **one** solution to this initial value problem. Therefore $\vec{z}(t) = \vec{w}(t)$ for all $\alpha < t < \beta$. It follows that

$$\vec{z}(t) = c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + \cdots + c_n\vec{y}_n(t),$$

and therefore $\vec{z}(t)$ is in the span of $\{\vec{y}_1(t), \vec{y}_2(t), \dots, \vec{y}_n(t)\}$. Since $z(t)$ was an arbitrary solution, this shows that $\{\vec{y}_1(t), \vec{y}_2(t), \dots, \vec{y}_n(t)\}$ spans the space of solutions.

Problem 9 *Wronskian Issues*

Let $A(t) = (a_{ij}(t))$ be an $n \times n$ matrix, with the functions $a_{ij}(t)$ continuous on the interval (α, β) for all i, j. Consider the differential equation

$$\vec{y}'(t) = A(t)\vec{y}(t).$$

Recall that the Wronskian $W[\vec{y}_1(t), \dots, \vec{y}_n(t)]$ of solutions $\vec{y}_1(t), \dots, \vec{y}_n(t)$ is nonzero on (α, β) if and only if the solutions are linearly independent. It's *very important* here that the functions we are considering are solutions to the differential equation $\vec{y}'(t) = A\vec{y}(t)$. To demonstrate this, consider the following functions

$$\vec{y}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}, \quad \vec{y}_2(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}$$

(a) Show that $W[\vec{y}_1(t), \vec{y}_2(t)]$ is identically 0

(b) Despite this, show that $\vec{y}_1(t)$ and $\vec{y}_2(t)$ are actually linearly independent

. . . .

Solution 9.

(a) We calculate

$$W[\vec{y}_1(t), \vec{y}_2(t)] = \det \begin{pmatrix} 1 & e^t \\ t & te^t \end{pmatrix} = te^t - te^t = 0.$$

(b) Suppose that c_1, c_2 are constants satisfying

$$c_1 \begin{pmatrix} 1 \\ t \end{pmatrix} + c_2 \begin{pmatrix} e^t \\ te^t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This would imply that $c_1 + c_2 e^t = 0$. If $c_2 \neq 0$, then this means that $c_1/c_2 = e^t$. However this is impossible, since c_1/c_2 is constant, and e^t is a function of t. Therefore $c_2 = 0$. This means that $c_1 = 0$ also, and this proves that $\vec{y}_1(t)$ and $\vec{y}_2(t)$ are linearly independent.