MATH 309: Homework #2

Due on: April 17, 2017

Problem 1 *Jordan Normal Form*

For each of the following values of the matrix A, find an invertible matrix P and a matrix N in Jordan normal form such that $P^{-1}AP = N$.

(a)
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(b)
$$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

(c)
$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(d)
$$A = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

(e)
$$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

(f)
$$A = \begin{pmatrix} 0 & 0 & 24 \\ 1 & 0 & 2 \\ 0 & 1 & -5 \end{pmatrix}$$

(g)
$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{pmatrix}$$

(h)
$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

(i)
$$A = \begin{pmatrix} -1 & -1 & 0 \\ 4 & 3 & 0 \\ -6 & -3 & 1 \end{pmatrix}$$

. . . .

Solution 1.

(a) The matrix A is already in Jordan normal form, so we can take $P = I$ and $N = A$.

(b) The characteristic polynomial is $p_A(x) = x^2 - 3x + 3$. The eigenvalues are therefore $(3/2) \pm i\sqrt{3}/2$. Therefore the matrix is diagonalizable, eg. its Jordan normal form N will be a diagonal matrix. Finding an eigenvector for each eigenvalue, we get

$$P = \begin{pmatrix} (-1/2) + i\sqrt{3}/2 & (-1/2) - i\sqrt{3}/2 \\ 1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} (3/2) + i\sqrt{3}/2 & 0 \\ 0 & (3/2) - i\sqrt{3}/2 \end{pmatrix}$$

(c) The characteristic polynomial is $p_A(x) = x^2-2x+2$. The eigenvalues are therefore $1 \pm i$. Therefore the matrix is diagonalizable, eg. its Jordan normal form N will be a diagonal matrix. Finding an eigenvector for each eigenvalue, we get

$$P = \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

(d) The charcteristic polynomial is $p_A(x) = x^2+2x-1$. The eigenvalues are therefore $-1 \pm \sqrt{2}$. Therefore the matrix is diagonalizable, eg. its Jordan normal form N will be a diagonal matrix. Finding an eigenvector for each eigenvalue, we get

$$P = \begin{pmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} -1+\sqrt{2} & 0 \\ 0 & -1-\sqrt{2} \end{pmatrix}$$

(e) The characteristic polynomial is $p_A(x) = x^2 + 2x + 1$. The eigenvalue is -1 with algebraic multiplicity two. However the geometric multiplicity is one and so N will be a 2×2 Jordan block $N = J_2(-1)$. Note that *any* nonzero vector in $\mathbb{R}^2$ will be a generalized eigenvector, since $(A - (-1)I)^2 = 0$. Note that $\vec{v} = \binom{1}{0}$ is not an eigenvector of A, and so it must be a generalized eigenvector of rank 2. This means that

$$\vec{w} := (A - (-1)I)\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector of A with eigenvalue -1. Thus we may take

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad N = J_2(-1) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

(f) The characteristic polynomial of this matrix is $p_A(x) = -x^3 - 5x^2 + 2x + 24 = -(x-2)(x+3)(x+4)$. The eigenvalues are therefore $2, -3, -4$. The matrix is therefore diagonalizable, eg. its Jordan form N will be a diagonal matrix. Finding an eigenvector for each eigenvalue, we get

$$P = \begin{pmatrix} 12 & -8 & -6 \\ 7 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, N = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

(g) The characteristic polynomial of this matrix is $p_A(x) = -x^3 - 3x^2 - 3x - 1 = -(x+1)^3$. The eigenvalue is therefore -1 with algebraic multiplicity 3. We calculate the corresponding eigenspace

$$E_{-1}(A) = N(A - (-1)I)) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

Therefore the geometric multiplicity is 1. It follows that the Jordan normal form of A is a 3×3 Jordan block $N = J_3(-1)$. We find a generalized eigenvector of rank 2 by solving

$$(A - (-1)I)\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

A solution is

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We also need a generalized eigenvector of rank 3 in this case, which we obtain by solving the equation

$$(A - (-1)I)\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

A solution is

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Putting this all together, we have

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, N = J_3(-1) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

(h) The characteristic polynomial is $p_A(x) = -x^3 + 3x - 2 = (x-1)^2(x+2)$. We calculate the corresponding eigenspaces

$$E_{-2}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}$$

$$E_1(A) = \text{span}\left\{\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}\right\}$$

In particular, this shows that 1 has algebraic multiplicity two but geometric multiplicity 1, and so we must still find a generalized eigenvector of rank two with eigenvalue 1. We can do this by solving the equation

$$(A - (1)I)\vec{v} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

A solution is given by

$$\vec{v} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}.$$

Putting this all together we have

$$P = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 1 & -1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) The characteristic polynomial of A is $p_A(x) = -(x-1)^3$, and therefore we have the eigenvalue 1 with algebraic multiplicity three. Furthermore, we calculate the eigenspace

$$E_1(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Therefore the geometric multiplicity of 1 is two. This means that we need to find a generalized eigenvector of rank 2. How can we do this? We can try to solve

$$(A - (1)I)\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

but we find that this has *no solution*. Similarly can try to solve

$$(A - (1)I)\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

but again this has *no solution*. What's the deal? The answer is we need to choose the "right" basis for the eigenspace $E_1(A)$. The way to do this is to find a generalized eigenvector first, and then find the regular eigenvectors after. Finding a generalized eigenvector is easy, actually. One may check that $(A - (1)I)^2 = 0$, and therefore every nonzero vector in $\mathbb{R}^3$ is a generalized eigenvector of A with eigenvalue λ. Choose any one that is not already an eigenvector of A, say

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then we know that $\vec{v}$ is a generalized eigenvector of A, and therefore must have rank 2. Now if we define $\vec{w}$ by

$$\vec{w} := (A - (1)I)\vec{v} = \begin{pmatrix} -2 \\ 4 \\ -6 \end{pmatrix}$$

then $\vec{w}$ is an eigenvector of A with eigenvalue 1. Finally if we choose any other eigenvector $\vec{e}$ to complete a basis for $E_1(A)$ (e.g. $\vec{e} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ but it doesn't matter which), we have the three vectors which will work as the column vectors for P:

$$P = (\vec{e}\ \vec{v}\ \vec{w}) = \begin{pmatrix} -2 & 1 & 0 \\ 4 & 0 & 0 \\ -6 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note here that the positioning of the column vectors in P is delicate and important. The generalized eigenvectors corresponding to a particular Jordan block need to be positioned in order of increasing rank.

Problem 2 Matrix Exponential

For each of the values of the matrix A in the previous problem, determine the value of $\exp(At)$

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Solution 2.

(a) We have the same eigenvalue 1, repeated twice, so

$$\exp(At) = e^t(I + (A - I)t) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

(b) We have two complex eigenvalues $a \pm ib$ for $a = 3/2$ and $b = \sqrt{3}/2$, and therefore

$$\begin{aligned} \exp(At) &= e^{(3/2)t}\cos((\sqrt{3}/2)t)I + e^{(3/2)t}\frac{2}{\sqrt{3}}(A - (3/2)I)\sin((\sqrt{3}/2)t) \\ &= \begin{pmatrix} e^{(3/2)t}\cos((\sqrt{3}/2)t) + (-1/\sqrt{3})e^{(3/2)t}\sin((\sqrt{3}/2)t) & (-5/\sqrt{3})e^{(3/2)t}\sin((\sqrt{3}/2)t) \\ (-1/\sqrt{3})e^{(3/2)t}\sin((\sqrt{3}/2)t) & e^{(3/2)t}\cos((\sqrt{3}/2)t) + (1/\sqrt{3})e^{(3/2)t}\sin((\sqrt{3}/2)t) \end{pmatrix} \end{aligned}$$

(c) We have two complex eigenvalues $a \pm ib$ for $a = 1$ and $b = 1$, and therefore

$$\exp(At) = e^t\cos(t)I + e^t(A - (1)I)\sin(t) = \begin{pmatrix} e^t\cos(t) & e^t\sin(t) \\ -e^t\sin(t) & e^t\cos(t) \end{pmatrix}$$

(d) The eigenvalues are real and distinct, given by $r_1 = -1 + \sqrt{2}$ and $r_2 = -1 - \sqrt{2}$ and therefore

$$\begin{aligned} \exp(At) &= \frac{1}{2\sqrt{2}}e^{(-1+\sqrt{2})t}(A - (-1-\sqrt{2})I) - \frac{1}{2\sqrt{2}}e^{(-1-\sqrt{2})t}(A - (-1+\sqrt{2})I) \\ &= \frac{1}{2\sqrt{2}}\begin{pmatrix} (1+\sqrt{2})e^{(-1+\sqrt{2})t} - (1-\sqrt{2})e^{(-1-\sqrt{2})t} & e^{(-1+\sqrt{2})t} - e^{(-1-\sqrt{2})t} \\ e^{(-1+\sqrt{2})t} - e^{(-1-\sqrt{2})t} & (-1+\sqrt{2})e^{(-1+\sqrt{2})t} - (-1-\sqrt{2})e^{(-1-\sqrt{2})t} \end{pmatrix} \end{aligned}$$

(e) The eigenvalue of A is -1 repeated twice. Therefore

$$\exp(At) = e^{-t}(I + (A - (-1)I)t) = \begin{pmatrix} e^{-t} + te^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{pmatrix}$$

(f) The eigenvalues of A are $r_1 = 2, r_2 = -3, r_3 = -4$. These are all distinct, so by Sylvester's formula we have that

$$\begin{aligned}\exp(At) &= e^{r_1 t}\frac{1}{(r_1 - r_2)(r_1 - r_3)}(A - r_2 I)(A - r_3 I)\\ &+ e^{r_2 t}\frac{1}{(r_2 - r_1)(r_2 - r_3)}(A - r_1 I)(A - r_3 I)\\ &+ e^{r_3 t}\frac{1}{(r_3 - r_1)(r_3 - r_2)}(A - r_1 I)(A - r_2 I)\\ &= \frac{1}{30}e^{2t}(A + 3I)(A + 4I) - \frac{1}{5}e^{-3t}(A - 2I)(A + 4I) + e^{-4t}\frac{1}{6}(A - 2I)(A + 3I)\end{aligned}$$

Calculating this we get the matrix

$$\begin{pmatrix} (12/30)e^{2t} + (8/5)e^{-3t} - (6/6)e^{-4t} & (24/30)e^{2t} - (24/5)e^{-3t} + (24/6)e^{-3t} & (48/30)e^{2t} + (72/5)e^{-3t} - (96/6)e^{- \\ (7/30)e^{2t} - (2/5)e^{-3t} + (1/6)e^{-4t} & (14/30)e^{2t} + (6/5)e^{-3t} - (4/6)e^{-3t} & (28/30)e^{2t} - (18/5)e^{-3t} + (16/6)e^{- \\ (1/30)e^{2t} - (1/5)e^{-3t} + (1/6)e^{-4t} & (2/30)e^{2t} + (3/5)e^{-3t} - (4/6)e^{-3t} & (4/30)e^{2t} - (9/5)e^{-3t} + (16/6)e^{-3t} \end{pmatrix}$$

(g) The eigenvalues of this matrix are all -1 (repeated three times). Therefore

$$\exp(At) = e^{-t}(I + (A - (-1)I)t + \frac{1}{2}(A - (-1)I)^2 t^2) = e^{-t}\begin{pmatrix} 1 + t + \frac{1}{2}t^2 & -\frac{1}{2}t^2 & -t + \frac{1}{2}t^2 \\ t + t^2 & 1 + t - t^2 & -3t + t^2 \\ \frac{1}{2}t^2 & t - \frac{1}{2}t^2 & 1 - 2t + \frac{1}{2}t^2 \end{pmatrix}$$

(h) The eigenvalues of this matrix are -2, and 1 twice repeated. We calculate the matrix exponential using the Jordan normal form found in Problem 1 part (h). To remind ourselves, $P^{-1}AP = N$ with

$$P = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 1 & -1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

From this we have

$$\exp(At) = P\exp(Nt)P^{-1}$$

where

$$\exp(Nt) = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}$$

Therefore since

$$P^{-1} = \begin{pmatrix} 1/9 & -2/9 & 4/9 \\ 4/9 & 1/9 & -2/9 \\ -3/9 & -3/9 & -3/9 \end{pmatrix}$$

the answer is $P\exp(Nt)P^{-1}$, the calculation of which we leave to the reader.

(i) In this case the eigenvalues of A are 1 (repeated three times). Therefore

$$\exp(At) = e^{-t}(I + (A - (1)I)t + \frac{1}{2}(A - (1)I)^2 t^2) = e^t\begin{pmatrix} 1 - 2t & -t & 0 \\ 4t & 1 + 2t & 0 \\ -6t & -3t & 1 \end{pmatrix}$$

Problem 3 Fundamental Matrix

Find a fundamental matrix for each of the following systems of equations

(a)

$$\begin{aligned} x' &= x + y \\ y' &= x - y \end{aligned}$$

(b)

$$\begin{aligned} x' &= -x - 4y \\ y' &= x - y \end{aligned}$$

(c)

$$\begin{aligned} x' &= x + y \\ y' &= 4x - 2y \end{aligned}$$

(d)

$$\begin{aligned} x' &= -x - 4y \\ y' &= x - y \end{aligned}$$

(e)

$$\begin{aligned} x' &= x - y \\ y' &= 5x - 3y \end{aligned}$$

(f)

$$\begin{aligned} x' &= 3x - 4y \\ y' &= x - y \end{aligned}$$

(g)

$$\begin{aligned} x' &= 4x - 8y \\ y' &= 8x - 4y \end{aligned}$$

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Solution 3.

(a) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are $\pm\sqrt{2}$. Therefore we get that

$$\Psi(t) = \exp(At) = \frac{1}{2\sqrt{2}} \begin{pmatrix} (1+\sqrt{2})e^{\sqrt{2}t} - (1-\sqrt{2})e^{-\sqrt{2}t} & e^{\sqrt{2}t} - e^{-\sqrt{2}t} \\ e^{\sqrt{2}t} - e^{-\sqrt{2}t} & (-1+\sqrt{2})e^{\sqrt{2}t} - (-1-\sqrt{2})e^{-\sqrt{2}t} \end{pmatrix}$$

(b) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are $-1 \pm 2i$. Therefore we get that

$$\Psi(t) = \exp(At) = \begin{pmatrix} e^{-t}\cos(2t) & -2e^{-t}\sin(2t) \\ (1/2)e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{pmatrix}$$

(c) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are 2 and -3. Therefore we get that

$$\Psi(t) = \exp(At) = \frac{1}{5} \begin{pmatrix} 4e^{2t} + e^{-3t} & e^{2t} - e^{-3t} \\ 4e^{2t} - 4e^{-3t} & e^{2t} + 4e^{-3t} \end{pmatrix}$$

(d) This is a repeat of (b)

(e) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are $-1 \pm i$. Therefore we get that

$$\Psi(t) = \exp(At) = \begin{pmatrix} e^{-t}\cos(t) + 2e^{-t}\sin(t) & -e^{-t}\sin(t) \\ 5e^{-t}\sin(t) & e^{-t}\cos(t) - 2e^{-t}\sin(t) \end{pmatrix}$$

(f) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are 1, repeated twice. Therefore

$$\Psi(t) = \exp(At) = e^t(I + (A - I)t) = e^t \begin{pmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t - 2te^t \end{pmatrix}$$

(g) We write the equation in the form $\vec{y}'(t) = A\vec{y}(t)$. The eigenvalues are $\pm 4\sqrt{3}$. Therefore

$$\Psi(t) = \exp(At) = \frac{1}{8\sqrt{3}} \begin{pmatrix} (4 + 4\sqrt{3})e^{4\sqrt{3}t} - (4 - 4\sqrt{3})e^{-4\sqrt{3}t} & -8e^{4\sqrt{3}t} + 8e^{-4\sqrt{3}t} \\ 8e^{4\sqrt{3}t} - 8e^{-4\sqrt{3}t} & (-4 + 4\sqrt{3})e^{4\sqrt{3}t} - (-4 - 4\sqrt{3})e^{-4\sqrt{3}t} \end{pmatrix}$$

Problem 4 Uniqueness of Fundamental Matrix

Let $A(t)$ be a matrix continuous on the interval (α, β). Show that if $\Psi(t)$ and $\Phi(t)$ are two fundamental matrices for the equation

$$\vec{y}'(t) = A(t)\vec{y}(t)$$

on the interval (α, β), then there exists a (constant) invertible matrix P so that $\Phi(t) = \Psi(t)P$.

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Solution 4. This is a tricky problem again – its solution will also be **extra credit**. Let $\Psi(t)$ and $\Phi(t)$ be two fundamental matrices for the equation, and choose $t_0 \in (\alpha, \beta)$. Set $P = \Psi(t_0)^{-1}\Phi(t_0)$. Then $\Phi(t_0) = \Psi(t_0)P$. Moreover, the column vectors of $\Phi(t)$ and $\Psi(t)$ are solutions to $\vec{y}'(t) = A(t)\vec{y}(t)$ (because they must form a fundamental set of solutions). The value of the first column of $\Phi(t)$ at $t = t_0$ agrees with the value of the first column of $\Psi(t)P$ at $t = t_0$. Therefore they both satisfy the same initial value problem, and by the Existence and Uniqueness Theorem this guarantees that they are equal for all t in the interval (α, β). The same argument in fact applies to the second column of each fundamental matrix, as well as the third, etc. Therefore $\Phi(t) = \Psi(t)P$ for all values of t.