MATH 309: Homework #3

Due on: May 1, 2017

Problem 1 *A 2×2 Homogeneous Equation with Complex Eigenvalues*

Without using matrix exponentials, find a fundamental set of solutions for the system of equations

$$\frac{d}{dx}\vec{y} = A\vec{y}, \quad A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}.$$

[Remember: the real part and the imaginary part of a solution is also a solution!]

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Problem 2 *Stability of the Origin I*

Consider the matrix $A = \begin{pmatrix} c & 1 \\ 1 & 2 \end{pmatrix}$. For each value of c, classify the stability of the critical point at the origin for the equation

$$\frac{d}{dx}\vec{y} = A\vec{y}.$$

. . . .

Problem 3 *Stability of the Origin II*

Consider the matrix $A = \begin{pmatrix} c & 1 \\ -1 & 2 \end{pmatrix}$. For each value of c, classify the stability of the critical point at the origin for the equation

$$\frac{d}{dx}\vec{y} = A\vec{y}.$$

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Problem 4 *Nonhomogeneous Equations I*

Determine the solution of the initial value problem

$$\frac{d}{dx}\vec{y} = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \quad \vec{b}(x) = \begin{pmatrix} e^{2x} \\ 0 \end{pmatrix}, \ \vec{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

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Problem 5 *Nonhomogeneous Equations II*

Determine the solution of the initial value problem

$$\frac{d}{dx}\vec{y} = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \quad \vec{b}(x) = \begin{pmatrix} e^{4x} \\ 0 \end{pmatrix}, \ \vec{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

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Problem 6 *Matrices with One Eigenvalue*

Let A be a 2×2 matrix, and suppose that A has exactly one eigenvalue λ with algebraic multiplicity 2. In this problem, we will show that

$$\exp(Ax) = Ie^{\lambda x} + (A - \lambda I)xe^{\lambda x} \tag{1}$$

(a) Define the matrix $N = (A - \lambda I)$. Show that $N^2 = 0$.

(b) Show that since $N^2 = 0$, we have $\exp(Nx) = I + Nx$

(c) Complete the proof of Equation (1) by using Proposition (1) below.

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Proposition 1. Suppose that B, C are two $n \times n$ matrices which commute, ie. $AB = BA$. Then

$$\exp(Ax + Bx) = \exp(Ax)\exp(Bx).$$

Problem 7 *Nonhomogeneous Equations Additional Practice*

(**NOT GRADED**) For each of the following, find the general solution.

(a)

$$\begin{cases} x' &= 2x - y + e^t \\ y' &= 3x - 2y + t \end{cases}$$

(b)

$$\begin{cases} x' &= x + y + e^{-2t} \\ y' &= 4x - 2y - 2e^t \end{cases}$$

(c)

$$\begin{cases} x' &= 2x - 5y - \cos(t) \\ y' &= x - 2y + \sin(t) \end{cases}$$

(d)

$$\begin{cases} x' &= -4x + 2y + t^3 \\ y' &= 2x - y - t^{-2} \end{cases}$$

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