MATH 309: Homework #6

Due on: June 2, 2017

Problem 1 *Heat Equation 1*

Find the solution of the heat conduction problem

$$\begin{aligned} 100u_{xx} &= u_t, \quad 0 < x < 1, \ t > 0 \\ u(0,t) &= u(1,t) = 0, \ t > 0 \\ u(x,0) &= \sin(2\pi x) - \sin(5\pi x) \end{aligned}$$

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Problem 2 *Heat Equation 2*

Find the solution of the heat conduction problem

$$\begin{aligned} u_{xx} &= 4u_t, \quad 0 < x < 2, \ t > 0 \\ u(0,t) &= u(2,t) = 0, \ t > 0 \\ u(x,0) &= 2\sin(\pi x/2) - \sin(\pi x) + 4\sin(2\pi x) \end{aligned}$$

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Problem 3 *Insulated Heat Equation Problem*

Consider a uniform rod of length L with an initial temperature given by $u(x,0) = \sin(\pi x/L)$ with $0 \le x \le L$. Assume that both ends of the bar are insulated (this is a homogeneous Neumann boundary condition for $t > 0$).

(a) Find the temperature $u(x,t)$. (Note: the initial condition $u(x,0)$ does not satisfy the boundary conditions, which is fine since we are only asking the boundary conditions to be satisfied for $t > 0$)

(b) What is the steady state temperature as $t \to \infty$?

(c) Let $\kappa^2 = 1$ and $L = 40$. Plot u vs. x for several values of t.

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Problem 4 *Another Insulated Heat Equation Problem*

Consider a bar of length 40 cm whose initial temperatore is given by $u(x,0) = x(60-x)/30$. Suppose that $\kappa^2 = 1/4$ cm^2/s and that both ends of the bar are insulated.

(a) Find the temperature $u(x,t)$. (Note: the initial condition $u(x,0)$ does not satisfy the boundary conditions, which is fine since we are only asking the boundary conditions to be satisfied for $t > 0$)

(b) What is the steady state temperature as $t \to \infty$?

(c) Plot u vs. x for several values of t.

(d) Determine how much time must elapse before the temperature at $x = 40$ comes within 1 degrees C of its steady state value.

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Problem 5 *Schrödinger Equation*

In quantum mechanics, the position of a point particle in space is not certain – it's described by a probability distribution. The probability distribution of the position of the particle is $|\psi(x,t)|^2$, where $\psi(x,t)$ is the **wave function** of the particle. (Note: the wave function $\psi(x,t)$ can be complex-valued!!). The one-dimensional, time-dependent Schrödinger equation, describing the wave function $\psi(x,t)$ of a particle of mass m interacting with a potential $v(x)$ is given by

$$i\hbar\psi_t(x,t) = -\frac{\hbar^2}{2m}\psi_{xx}(x,t) + v(x)\psi(x,t)$$

where $\hbar$ is some universal constant. The potential $v(x)$ can be imagined as a function describing the particles interaction with whatever "stuff" is in the space surrounding the particle, eg. walls, external forces, etc.

(a) Use separation of variables to replace this partial differential equation with a pair of two ordinary differential equations

(b) If $v(x)$ is a potential corresponding to an "infinite square well":

$$v(x) = \begin{cases} 0, & -1 < x < 1 \\ \infty, & |x| \geq 1 \end{cases}$$

Then $\psi(x,t)$ must be zero whenever $|x| \geq 1$ and therefore $\psi(x,t)$ is the wave function of a particle trapped in a one-dimensional box! In other words, this potential describes a particle surrounded by impermeable walls. In this case, Schrödinger's equation reduces to

$$\begin{aligned} i\hbar\psi_t(x,t) &= -\frac{\hbar^2}{2m}\psi_{xx}(x,t), \quad -1 < x < 1,\ t > 0 \\ \psi(-1,t) &= \psi(1,t) = 0,\ t > 0 \end{aligned}$$

Suppose that initially the wave function is known to be

$$\psi(x,0) = \frac{3}{5}\sin(\pi x) + \frac{4}{5}\sin(3\pi x).$$

Determine $\psi(x,t)$ for all $t > 0$.

(c) Since $|\psi(x,t)|^2$ is the probability *distribution* of the particle's position at time t, the probability that the particle is somewhere in the box between ℓ_1 and ℓ_2 is given by

$$\mathbb{P}(\ell_1 \le \text{pos} \le \ell_2) = \int_{\ell_1}^{\ell_2} |\psi(x,t)|^2 dx.$$

Show that the probability $\mathbb{P}(-1 \le \text{pos} \le 1)$ that the particle is between -1 and 1 is always 1 (in other words, the particle is always in the box!).

(d) What is the probability $\mathbb{P}(-1 \le \text{pos} \le 0)$ that the particle is in the first half of the box at any given time?

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