MATH 309: Homework #6

Due on: May 25, 2016

Problem 1 *Heat Equation 1*

Find the solution of the heat conduction problem

$$100u_{xx} = u_t, \quad 0 < x < 1, \ t > 0$$
$$u(0,t) = u(1,t) = 0, \ t > 0$$
$$u(x,0) = \sin(2\pi x) - \sin(5\pi x)$$

. . . .

Solution 1. We identify $\alpha^2 = 100$ and $L = 1$. Then we need to expand $u(x,0)$ as

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

However, if we look at the form of $u(x,0)$, this is immediately accomplished by taking $b_2 = 1, b_5 = -1$ and $b_n = 0$ otherwise. Therefore

$$u(x,t) = u_2(x,t) - u_5(x,t) = e^{-400\pi^2 t}\sin(2\pi x) - e^{-2500\pi^2 t}\sin(5\pi x).$$

Problem 2 *Heat Equation 2*

Find the solution of the heat conduction problem

$$u_{xx} = 4u_t, \quad 0 < x < 2, \ t > 0$$
$$u(0,t) = u(2,t) = 0, \ t > 0$$
$$u(x,0) = 2\sin(\pi x/2) - \sin(\pi x) + 4\sin(2\pi x)$$

. . . .

Solution 2. We identify $\alpha^2 = 4$ and $L = 2$. Then we need to expand $u(x,0)$ as

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/2).$$

However, if we look at the form of $u(x,0)$, this is immediately accomplished by taking $b_1 = 2, b_2 = -1, b_4 = 4$ and $b_n = 0$ otherwise. Therefore

$$u(x,t) = u_1(x,t) - u_2(x,t) + 4u_4(x,t) = e^{-\pi^2 t}\sin(\pi x/2) - e^{-4\pi^2 t}\sin(\pi x) + 4e^{-16\pi^2 t}\sin(2\pi x).$$

Problem 3 *Insulated Heat Equation Problem*

Consider a uniform rod of length L with an initial temperature given by $u(x,0) = \sin(\pi x/L)$ with $0 \le x \le L$. Assume that both ends of the bar are insulated (this is a homogeneous Neumann boundary condition for $t > 0$).

(a) Find the temperature $u(x,t)$. (Note: the initial condition $u(x,0)$ does not satisfy the boundary conditions, which is fine since we are only asking the boundary conditions to be satisfied for $t > 0$)

(b) What is the steady state temperature as $t \to \infty$?

(c) Let $\alpha^2 = 1$ and $L = 40$. Plot u vs. x for several values of t.

. . . .

Solution 3.

(a) We need to determine the temperature initially in terms of a cosine series. This means reflecting $\sin(\pi x/L)$ *evenly* and then extending periodically. In other words, we're really looking for the cosine series of $|\sin(\pi x/L))|$. Using Euler-Fourier, we obtain

$$a_n = \frac{1}{L}\int_{-L}^{L} |\sin(\pi x/L)| \cos(n\pi x/L)dx = \frac{2}{L}\int_0^L \sin(\pi x/L)\cos(n\pi x/L)dx.$$

Now in order to complete the last integral on the right, we can adopt several strategies. The most obvious thing is to integrate by parts twice, and then compare sides – however, that is a lot of work. A shorter strategy is to use the addition angle formulas for sine to write:

$$\sin(\pi x/L)\cos(n\pi x/L) = \frac{1}{2}(\sin((1+n)\pi x/L) + \sin((1-n)\pi x/L)).$$

With this in mind, the above integral becomes

$$a_n = \frac{1}{L}\int_0^L (\sin((1+n)\pi x/L) + \sin((1-n)\pi x/L))dx = \frac{2}{\pi}\left(\frac{1+(-1)^n}{1-n^2}\right).$$

However, notice that in our derivation of this formula, we divided by $1-n$, and therefore the expression we obtained for a_n does not apply when $n = 1$. We must treat this case separately! We calculate using the double angle formula

$$a_1 = \frac{2}{L}\int_0^L \sin(\pi x/L)\cos(\pi x/L)dx = \frac{1}{L}\int_0^L \sin(2\pi x/L)dx = -\frac{1}{2\pi}\cos(2\pi x/L)|_0^1 = 0.$$

We conclude that

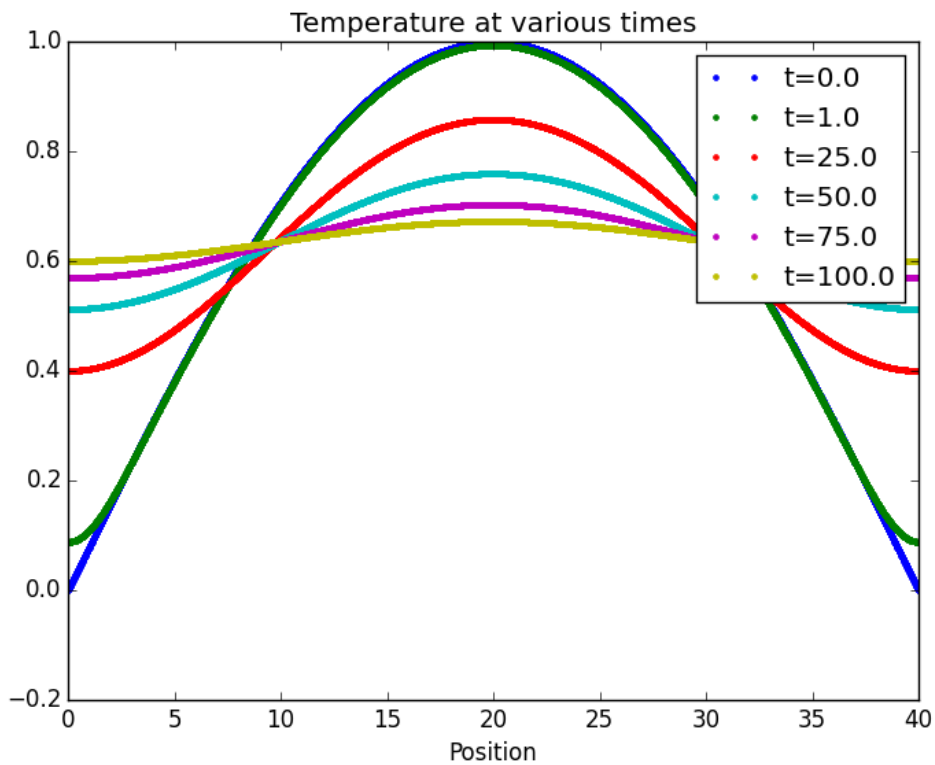
$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi}\left(\frac{1+(-1)^n}{1-n^2}\right)\cos(n\pi x/L).$$

This tells us that

$$u(x,t) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi} \left(\frac{1 + (-1)^n}{1 - n^2} \right) e^{-n^2\pi^2\alpha^2 t/L^2} \cos(n\pi x/L).$$

(b) As $t \to \infty$, the exponential terms die off, leaving only $a_0/2$. Therefore the steady state temperature is $2/\pi$.

(c) Plot at several times is included in the figure below.



Problem 4 *Another Insulated Heat Equation Problem*

Consider a bar of length 40 cm whose initial temperatore is given by $u(x,0) = x(60 - x)/30$. Suppose that $\alpha^2 = 1/4$ cm^2/s and that both ends of the bar are insulated.

(a) Find the temperature $u(x,t)$. (Note: the initial condition $u(x,0)$ does not satisfy the boundary conditions, which is fine since we are only asking the boundary conditions to be satisfied for $t > 0$)

(b) What is the steady state temperature as $t \to \infty$?

(c) Plot u vs. x for several values of t.

(d) Determine how much time must elapse before the temperature at $x = 40$ comes within 1 degrees C of its steady state value.

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Solution 4.

(a) Again, we must extend $u(x, 0)$ evenly and periodically in order to pick up its cosine series. Then by the Euler-Fourier equation we have

$$a_n = \frac{2}{40}\int_0^{40} \frac{x(60-x)}{30}\cos(n\pi x/40)dx.$$

We can obtain the value explicitly by using integration by parts twice to get the $a_n's$. (There are, of course, more clever ways to do things, but this works fine). Doing so, we obtain

$$a_n = \frac{160}{3}\frac{(-1)^{n+1}-3}{n^2\pi^2},$$

which works except for $n = 0$, for which we obtain $a_0 = 400/9$. Therefore we see

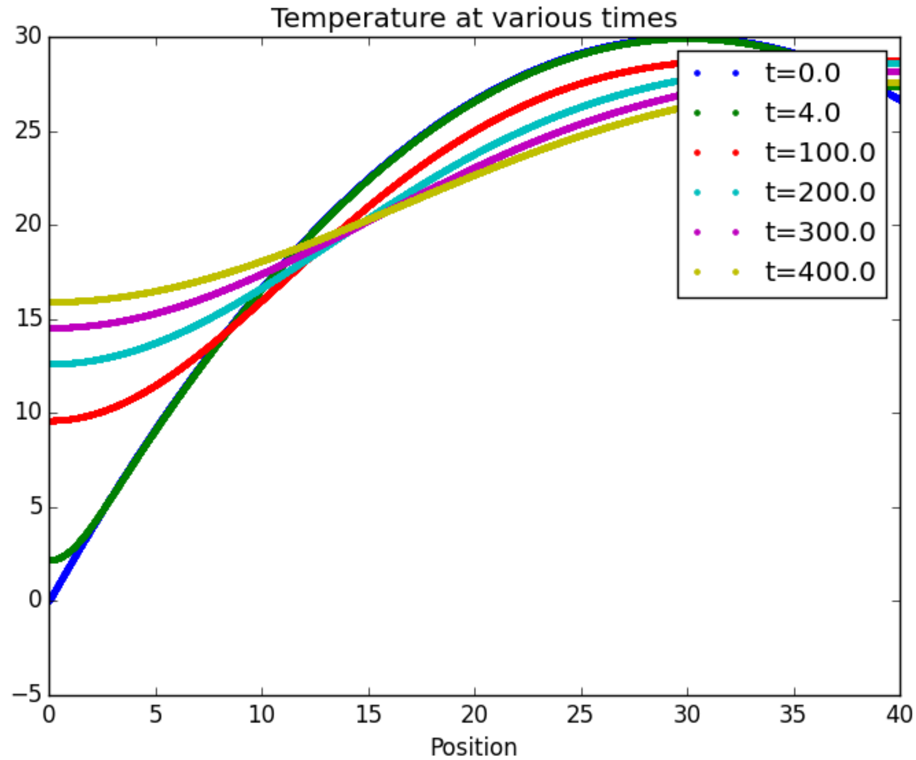
$$u(x,0) = 200/9 + \sum_{n=1}^{\infty} \frac{160}{3}\frac{(-1)^{n+1}-3}{n^2\pi^2}\cos(n\pi x/40).$$

We conclude that

$$u(x,t) = 200/9 + \sum_{n=1}^{\infty} \frac{160}{3}\frac{(-1)^{n+1}-3}{n^2\pi^2}e^{-n^2\pi^2 t/6400}\cos(n\pi x/40).$$

(b) Again, the exponential terms die off, so the steady state temperature is 200/9.

(c) Plot at several times is included in the figure below.



Problem 5 *Schrödinger Equation*

In quantum mechanics, the position of a point particle in space is not certain – it's described by a probability distribution. The probability distribution of the position of the particle is $|\psi(x,t)|^2$, where $\psi(x,t)$ is the **wave function** of the particle. (Note: the wave function $\psi(x,t)$ can be complex-valued!!). The one-dimensional, time-dependent Schrödinger equation, describing the wave function $\psi(x,t)$ of a particle of mass m interacting with a potential $v(x)$ is given by

$$i\hbar\psi_t(x,t) = -\frac{\hbar^2}{2m}\psi_{xx}(x,t) + v(x)\psi(x,t)$$

where $\hbar$ is some universal constant. The potential $v(x)$ can be imagined as a function describing the particles interaction with whatever "stuff" is in the space surrounding the particle, eg. walls, external forces, etc.

(a) Use separation of variables to replace this partial differential equation with a pair of two ordinary differential equations

(b) If $v(x)$ is a potential corresponding to an "infinite square well":

$$v(x) = \begin{cases} 0, & -1 < x < 1 \\ \infty, & |x| \geq 1 \end{cases}$$

Then $\psi(x,t)$ must be zero whenever $|x| \geq 1$ and therefore $\psi(x,t)$ is the wave function of a particle trapped in a one-dimensional box! In other words, this potential describes a particle surrounded by impermeable walls. In this case, Schrödinger's equation reduces to

$$i\hbar\psi_t(x,t) = -\frac{\hbar^2}{2m}\psi_{xx}(x,t), \quad -1 < x < 1, \ t > 0$$
$$\psi(-1,t) = \psi(1,t) = 0, \ t > 0$$

Suppose that initially the wave function is known to be

$$\psi(x,0) = \frac{3}{5}\sin(\pi x) + \frac{4}{5}\sin(3\pi x).$$

Determine $\psi(x,t)$ for all $t > 0$.

(c) Since $|\psi(x,t)|^2$ is the probability *distribution* of the particle's position at time t, the probability that the particle is somewhere in the box between ℓ_1 and ℓ_2 is given by

$$\mathbb{P}(\ell_1 \leq \text{pos} \leq \ell_2) = \int_{\ell_1}^{\ell_2} |\psi(x,t)|^2 dx.$$

Show that the probability $\mathbb{P}(-1 \leq \text{pos} \leq 1)$ that the particle is between -1 and 1 is always 1 (in other words, the particle is always in the box!).

(d) What is the probability $\mathbb{P}(-1 \leq \text{pos} \leq 0)$ that the particle is in the first half of the box at any given time?

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Solution 5.

(a) We assume $\psi(x,t) = F(x)G(t)$. Then inserting this into Schrödinger's equation, we obtain

$$i\hbar F(x)G'(t) = -\frac{\hbar^2}{2m}F''(x)G(t) + v(x)F(x)G(t).$$

Now if we divide out by a $G(t)$ and a $F(x)$ we find

$$i\hbar G'(t)/G(t) = -\frac{\hbar^2}{2m}F''(x)/F(x) + v(x).$$

The function on the left hand side is a function of t only. The function on the right hand side is a function of x only. Therefore the only way that the above equality can work is if both sides are equal to some constant E. Therefore

$$i\hbar G'(t)/G(t) = E, \quad -\frac{\hbar^2}{2m}F''(x)/F(x) + v(x) = E.$$

Simplifying, this gives us the system of two ordinary differential equations

$$i\hbar G'(t) = EG(t).$$

$$-\frac{\hbar^2}{2m}F''(x) + v(x)F(x) = EF(x).$$

The latter equation of these two equations is known as the **time-independent Schrödinger equation**.

(b) This is just like the heat equation, with $\alpha^2 = i\frac{\hbar}{2m}$ and $L = 1$. Thus given the initial condition, the solution that we are looking for is

$$\psi(x,t) = \frac{3}{5}e^{-i\frac{\hbar\pi^2}{2m}t}\sin(\pi x) + \frac{4}{5}e^{-i\frac{9\hbar\pi^2}{2m}t}\sin(3\pi x).$$

(c) Note that

$$\psi(x,t)^* = \frac{3}{5}e^{i\frac{\hbar\pi^2}{2m}t}\sin(\pi x) + \frac{4}{5}e^{i\frac{9\hbar\pi^2}{2m}t}\sin(3\pi x),$$

and therefore

$$|\psi(x,t)|^2 = \psi(x,t)\psi(x,t)^* = \frac{9}{25}\sin^2(\pi x)+\frac{16}{25}\sin^2(3\pi x)+\frac{12}{25}\left(e^{i\frac{8\hbar\pi^2}{2m}t} + e^{i\frac{-8\hbar\pi^2}{2m}t}\right)\sin(\pi x)\sin(3\pi x).$$

If we now integrate over the domain of the box (from -1 to 1), orthogonality tells us the integral of $\sin(\pi x)\sin(3\pi x)$ dies off! Therefore we obtain:

$$\int_{-1}^{1}|\psi(x,t)|^2dx = \frac{9}{25}\int_{-1}^{1}\sin^2(\pi x)dx + \frac{16}{25}\int_{-1}^{1}\sin^2(3\pi x)dx = \frac{9}{25} + \frac{16}{25} = 1.$$

This shows that the probability that the particle is in the box at any time t is 1 – e.g. it is a certainty.

(d) We can use the work from above to write

$$\int_{-1}^{1}|\psi(x,t)^2dx = \frac{9}{25}\int_{-1}^{0}\sin^2(\pi x)dx+\frac{16}{25}\int_{-1}^{0}\sin^2(3\pi x)dx+\frac{12}{25}\left(e^{i\frac{8\hbar\pi^2}{2m}t} + e^{i\frac{-8\hbar\pi^2}{2m}t}\right)\int_{-1}^{0}\sin(\pi x)\sin(3\pi$$

However, since we're not integrating over the full period, we cannot appeal to orthogonality to say that the cross-term dies anymore. However, direct calculation shows that it does indeed die anyway. The sum of the first two integrals is easily calculated to be 1/2. Therefore the probability that the particle is in the first half of the box at any time t is exactly 1/2. In other words – at any time the particle is equally likely to be in either side of the box.