Math 309 Lecture 11

Nonhomogeneous Equations

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Today!

Plan for today:

- Basic Theory
- Method of Undetermined Coefficients
- Method of Variation of Parameters

Next time:

- More Nonhomogeneous Differential Equations

Outline

Back in Math 307...

- Back in Math 307, we considered differential equations of the form
$$y'' + by' + cy = f(x).$$
- To find the general solution, we found a **particular solution** and added the general solution of the corresponding homogeneous equation.
$$y = y_p + y_h$$
- The same idea works here!

General Solution

Proposition

Suppose that $\vec{y}_1, \vec{y}_2$ are solutions of the nonhomogeneous system

$$\vec{y}' = A(x)\vec{y} + \vec{b}(x).$$

Then $\vec{y}_1 - \vec{y}_2$ is a solution of the corresponding homogeneous equation

$$\vec{y}_h' = A(x)\vec{y}_h.$$

- in other words, any two solutions to nonhomogeneous differ by a solution of homogeneous!
- this characterizes solutions to nonhomogeneous

General Solution

Proposition

If $\vec{y}_p$ is any single solution to the nonhomogeneous equation

$$\vec{y}' = A(x)\vec{y} + \vec{b}(x).$$

Then the general solution to the nonhomogeneous equations is

$$\vec{y} = \vec{y}_p + \vec{y}_h$$

where $\vec{y}_h$ is the general solution of the associated homogeneous equation

$$\vec{y}_h' = A(x)\vec{y}_h.$$

- y_p is called a **particular solution** (not unique!)

Back in Math 307

- for a second-order equation

$$y'' + 2y' + y = e^{3x}$$

- we'd propose a particular solution of the form $y_p = ce^{3x}$
- then we'd determine c by inserting our guess into the differential equation:

$$9ce^{3x} + 6ce^{3x} + ce^{3x} = e^{3x}.$$

$$9c + 6c + c = 1 \ \Rightarrow c = 1/16.$$

- this shows $y_p = (1/16)e^{3x}$ is a solution

Method of Undetermined Coefficients

- To find a particular solution of

$$\vec{y}' = A\vec{y} + e^{rx}\vec{v}, \quad (A, \vec{v}, r \text{ all constant})$$

- if r is **not an eigenvalue** of A, we propose a solution of the form

$$\vec{y}_p = \vec{c}e^{rx}$$

- plugging this into the equation, we get:

$$r\vec{c}e^{rx} = A\vec{c}e^{rx} + e^{rx}\vec{v}.$$

$$(A - rI)\vec{c} = -\vec{v} \quad \Rightarrow \quad \vec{c} = -(A - rI)^{-1}\vec{v}.$$

- then $\vec{y}_p = \vec{c}e^{rx}$ is a particular solution!

Method of Undetermined Coefficients

- More generally, to find a particular solution of

$$\vec{y}' = A\vec{y} + e^{rx}(\vec{v}_1 x + \vec{v}_0), \quad (A, \vec{v}, r \text{ all constant})$$

- if r is **not an eigenvalue** of A, we propose a solution of the form

$$\vec{y}_p = (\vec{c}_1 x + \vec{c}_0)e^{rx}$$

- plugging this into the equation, we get:

$$r\vec{c}_1 x e^{rx} + (\vec{c}_1 + r\vec{c}_0)e^{rx} = A\vec{c}_1 x e^{rx} + A\vec{c}_0 e^{rx} + e^{rx}\vec{v}_1 x + e^{rx}\vec{v}_0.$$

$$(A - rI)\vec{c}_1 = -\vec{v}_1 \quad \Rightarrow \quad \vec{c}_1 = -(A - rI)^{-1}\vec{v}_1.$$

$$(A - rI)\vec{c}_0 = -\vec{v}_1 + \vec{c}_1 \quad \Rightarrow \quad \vec{c}_0 = -(A - rI)^{-1}(\vec{v}_1 - \vec{c}_1).$$

- then $\vec{y}_p = (\vec{c}_1 x + \vec{c}_0)e^{rx}$ is a particular solution!

Example 1

Question

Find a particular solution to the differential equation

$$\vec{y}' = A\vec{y} + \begin{pmatrix} 2e^{2x} \\ e^{2x} \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

- we propose a solution of the form $\vec{y}_p = \vec{c}e^{2x}$
- then

$$\vec{c} = -(A - 2I)^{-1} = -\begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix}$$

- thus $\vec{y}_p = \binom{-4/3}{-5/3} e^{2x}$ is a particular solution

Example 2

Question

Find a particular solution to the differential equation

$$\vec{y}' = A\vec{y} + \begin{pmatrix} 2e^{2x} \\ e^{2x} + e^{x} \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

- to solve this equation, we split the equation into two new equations:

$$\vec{y}_1' = A\vec{y}_1 + \begin{pmatrix} 2e^{2x} \\ e^{2x} \end{pmatrix}, \quad \vec{y}_2' = A\vec{y}_2 + \begin{pmatrix} 1 \\ e^{x} \end{pmatrix}$$

- if $\vec{y}_{p1}$ and $\vec{y}_{p2}$ are particular solutions of each of these, then $\vec{y}_p = \vec{y}_{p1} + \vec{y}_{p2}$ is a solution of the original!

Example 2 Continued

Question

Find a particular solution to the differential equation

$$\vec{y}' = A\vec{y} + \begin{pmatrix} 2e^{2x} \\ e^{2x} + e^{x} \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

- from Example 1, $\vec{y}_{p1} = \binom{-4/3}{-5/3} e^{2x}$
- for $\vec{y}_{p2}$, we propose $\vec{y}_{p2} = \vec{c}_2 e^{x}$. then

$$\vec{c}_2 = -(A - I)^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}.$$

- therefore

$$\vec{y}_p = \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix} e^{2x} + \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} e^{x}.$$

Example 3

Question

Find a particular solution to the differential equation

$$\vec{y}' = A\vec{y} + \begin{pmatrix} xe^x \\ e^x \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

- we propose a solution of the form $\vec{y}_p = (\vec{c}_1 x + \vec{c}_0)e^x$
- then

$$\vec{c}_1 = -(A - I)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}, \quad \vec{c}_0 = -(A - I)^{-1}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}\right.$$

- therefore we have the particular solution

$$\vec{y}_p = \left(\begin{pmatrix} 0 \\ -1/2 \end{pmatrix} x + \begin{pmatrix} -3/4 \\ 0 \end{pmatrix}\right) e^x = \begin{pmatrix} -3/4 \\ -x/2 \end{pmatrix} e^x$$

Example 4

Question

Find a particular solution to the differential equation

$$\vec{y}' = A\vec{y} + \begin{pmatrix} e^{-x} \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

- we propose a solution of the form $\vec{y}_p = \vec{c}e^{-x}$. Then

$$\vec{c} = -(A+I)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \ ?????.$$

- the matrix $A + I$ is singular, so no inverse!
- this is because -1 is an eigenvalue of A – our method doesn't work for this!

- back in Math 307, you may have been exposed to a method called variation of parameters
- basic idea: to solve

$$y' = a(x)y + b(x),$$

- propose a solution of the form $y_p = v(x)y_h$, where y_h is a solution of homogeneous equation

$$y_h' = a(x)y_h$$

- then $v'(x) = b(x)/y_h(x)$, and so

$$y_p = y_h(x)\int \frac{b(x)}{y_h(x)}dx.$$

- we generalize this here!

The Method's Derivation

- Consider the nonhomogeneous equation

$$\vec{y}' = A(x)\vec{y} + \vec{b}(x).$$

- The associated homogeneous equation is:

$$\vec{y}_h' = A(x)\vec{y}_h.$$

- Let $\Phi(x)$ be a fundamental matrix for the homogeneous equation
- Propose $\vec{y}_p = \Phi(x)\vec{v}(x)$
- How can we find $v(x)$?

The Method's Derivation (Continued)

- We calculate:

$$\vec{y}_p' = (\Phi(x)\vec{v}(x))' = \Phi'(x)\vec{v}(x) + \Phi(x)\vec{v}'(x)$$

- Since $\Phi(x)$ is a fundamental matrix, $\Phi'(x) = A(x)\Phi(x)$, so:

$$\vec{y}_p' = A(x)\Phi(x)\vec{v}(x) + \Phi(x)\vec{v}'(x)$$

- Moreover

$$\vec{y}_p' = A(x)\vec{y}_p + \vec{b}(x) = A(x)\Phi(x)\vec{v}(x) + \vec{b}(x)$$

- therefore

$$A(x)\Phi(x)\vec{v}(x) + \vec{b}(x) = A(x)\Phi(x)\vec{v}(x) + \Phi(x)\vec{v}'(x).$$

The Method's Derivation (Continued)

- Simplifying:

$$\vec{b}(x) = \Phi(x)\vec{v}'(x).$$

- Thus

$$\vec{v}'(x) = \Phi(x)^{-1}\vec{b}(x).$$

- Therefore

$$\vec{v}(x) = \int \Phi(x)^{-1}\vec{b}(x)dx.$$

- and thus

$$\vec{y}_p(x) = \Phi(x)\vec{v}(x) = \Phi(x)\int \Phi(x)^{-1}\vec{b}(x)dx.$$

Method Summary

- to find a particular solution of

$$\vec{y}' = A(x)\vec{y}(x) + \vec{b}(x),$$

- find a fundamental matrix $\Phi(x)$ for $\vec{y}_h' = A(x)\vec{y}_h(x)$
- then we have

$$\vec{y}_p = \Phi(x) \int \Phi(x)^{-1}\vec{b}(x)dx.$$

- DOWNSIDE: this calculation can take a while...

Example 1

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}.$$

- We first calculate a fundamental matrix for $\vec{y}' = A\vec{y}$.
- The eigenvalues of A are $\pm i$
- A fundamental matrix is therefore $\Phi(x) = \exp(Ax)$, with

$$\exp(Ax) = \cos(x)I + \sin(x)A = \begin{pmatrix} \cos(x) + 2\sin(x) & -5\sin(x) \\ \sin(x) & \cos(x) - 2\sin(x) \end{pmatrix}.$$

Example 1 (Continued)

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}.$$

- Note: since $\exp(Ax)^{-1} = \exp(-Ax)$, we have $\Phi(x)^{-1} = \Phi(-x)$ and therefore

$$\Phi(x)^{-1}\begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix} = \begin{pmatrix} \cos(x) - 2\sin(x) & 5\sin(x) \\ -\sin(x) & \cos(x) + 2\sin(x) \end{pmatrix}\begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix}$$
$$= \begin{pmatrix} -\cos^2(x) + 2\sin(x)\cos(x) + 5\sin^2(x) \\ 2\sin(x)\cos(x) + 2\sin^2(x) \end{pmatrix} = \begin{pmatrix} -1 + 6\sin^2(x) + 2\sin(x)\cos(x) \\ 2\sin(x)\cos(x) + 2\sin^2(x) \end{pmatrix}.$$

Example 1 (Continued)

Question

Find a particular solution of the differential equation

$$\vec{y}\,' = A\vec{y} + \begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}.$$

- Integrating, we obtain

$$\int \Phi(x)^{-1}\begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix} dx = \int \begin{pmatrix} -1 + 6\sin^2(x) + 2\sin(x)\cos(x) \\ 2\sin(x)\cos(x) + 2\sin^2(x) \end{pmatrix} dx$$
$$= \begin{pmatrix} 2x - (3/2)\sin(2x) + \sin^2(x) \\ \sin^2(x) + x - (1/2)\sin(2x) \end{pmatrix}$$

Example 1 (Continued)

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}.$$

- Finally, we find

$$\begin{aligned}
\vec{y}_p &= \Phi(x) \int \Phi(x)^{-1} \begin{pmatrix} -\cos(x) \\ \sin(x) \end{pmatrix} dx \\
&= \begin{pmatrix} \cos(x) + 2\sin(x) & -5\sin(x) \\ \sin(x) & \cos(x) - 2\sin(x) \end{pmatrix} \begin{pmatrix} 2x - (3/2)\sin(2x) + \sin^2(x) \\ \sin^2(x) + x - (1/2)\sin(2x) \end{pmatrix} \\
&= \begin{pmatrix} 2x\cos(x) - x\sin(x) - 3\sin(x) \\ x\cos(x) - \sin(x) \end{pmatrix}
\end{aligned}$$

- the last step resulting from a massive fireball of trig identities

summary!

what we did today:

- method of undetermined coefficients
- method of variation of parameters

plan for next time:

- more nonhomogeneous differential equations