Math 309 Lecture 12

More Nonhomogeneous Equations

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Today!

Plan for today:

- Diagonalization Method

Next time:

- Fourier Series

Outline

Solving Nonhomogeneous Equations

TWO METHODS SO FAR:

(1) Method of Undetermined Coefficients

(2) Method of Variation of Parameters

- What if undetermined coefficients doesn't work?
- Variation of parameters ... but it takes so long!
- Alternative method:

(3) Method of Diagonalization

Method of Diagonalization

- to find a solution to the differential equation

$$\vec{y}' = A\vec{y} + \vec{b}(x)$$

- use the following steps:

STEP 1: find P so that $P^{-1}AP = N$ is in Jordan normal form

STEP 2: substitute $\vec{y} = P\vec{z}$, so the equation becomes:

$$(P\vec{z})' = AP\vec{z} + \vec{b}(x)$$

STEP 3: multiply by P^{-1}, obtaining

$$\vec{z}' = N\vec{z} + P\vec{b}(x).$$

STEP 4: solve for $\vec{z}$, and get final answer $\vec{y} = P\vec{z}$

Example 1

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \ \vec{b}(x) = \begin{pmatrix} e^{-x} \\ 0 \end{pmatrix}.$$

- -1 is an eigenvalue of A, so undetermined coefficients doesn't work
- don't want variation of parameters – too much work!!!
- instead try diagonalization!

Example 1 (Continued)

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \vec{b}(x) = \begin{pmatrix} e^{-x} \\ 0 \end{pmatrix}.$$

- usual calculation shows $P^{-1}AP = N$:

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

- sub $\vec{y} = P\vec{z}$:

$$\vec{z}' = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \vec{z} + P^{-1}\vec{b}(x)$$

Example 1 (Continued)

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \ \vec{b}(x) = \begin{pmatrix} e^{-x} \\ 0 \end{pmatrix}.$$

- simplifies:

$$\vec{z}' = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \vec{z} + \begin{pmatrix} e^{-x}/2 \\ -e^{-x}/2 \end{pmatrix}.$$

- Write $\vec{z} = \binom{z_1}{z_2}$. Then the above says:

$$z_1' = -z_1 + \frac{1}{2}e^{-x}, \qquad z_2' = 3z_2 - \frac{1}{2}e^{-x}.$$

- These are both first-order ODEs! Super easy to solve!!

Example 1 (Continued)

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \ \vec{b}(x) = \begin{pmatrix} e^{-x} \\ 0 \end{pmatrix}.$$

- we can use integrating factor method ...
- then we obtain solutions

$$z_1 = \frac{1}{2}xe^{-x}, \quad z_2 = \frac{1}{8}e^{-x}.$$

- therefore $\vec{z} = \binom{xe^{-x}/2}{e^{-x}/8}$ and finally

$$\vec{y} = P\vec{z} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\vec{z} = \begin{pmatrix} xe^{-x}/2 + e^{-x}/8 \\ -xe^{-x}/2 + e^{-x}/8 \end{pmatrix}.$$

Example 2

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ \vec{b}(x) = \begin{pmatrix} e^x \\ e^x \end{pmatrix}.$$

- again 1 is an eigenvalue for A, so undetermined coefficients is no good
- variation of parameters is still sooo much work
- instead, try diagonalization!

Example 2 (Continued)

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \vec{b}(x) = \begin{pmatrix} e^x \\ e^x \end{pmatrix}.$$

- usual calculation gives $P^{-1}AP = N$:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

- substitute $\vec{y} = P\vec{z}$:

$$\vec{z}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \vec{z} + P^{-1}\vec{b}(x)$$

Example 2 (Continued)

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \; \vec{b}(x) = \begin{pmatrix} e^x \\ e^x \end{pmatrix}.$$

- simplifies to

$$\vec{z}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \vec{z} + \begin{pmatrix} e^x \\ 2e^x \end{pmatrix}$$

- write $\vec{z} = \binom{z_1}{z_2}$. Then

$$z_1' = z_1 + z_2 + e^x, \qquad z_2' = z_2 + 2e^x.$$

- the second equation above is ordinary first order linear, easy to solve!!

Example 2 (Continued)

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \vec{b}(x), \quad A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ \vec{b}(x) = \begin{pmatrix} e^x \\ e^x \end{pmatrix}.$$

- using integrating factor, get $z_2 = 2xe^x$
- now substituting z_2 into equation for z_1, we find

$$z_1' = z_1 + 2xe^x + e^x.$$

- this is now ordinary first order linear, easy to solve!!
- integrating factor gives $z_1 = (x^2 + x)e^x$

Example 2 (Continued)

Question

Find a particular solution of the differential equation

$$\vec{y}' = A\vec{y} + \vec{b}(x), \;\; A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \; \vec{b}(x) = \begin{pmatrix} e^x \\ e^x \end{pmatrix}.$$

- this means that

$$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2xe^x \\ x^2e^x + xe^x \end{pmatrix}.$$

- therefore we have that

$$\vec{y} = P\vec{z} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 2xe^x \\ x^2e^x + xe^x \end{pmatrix} = \begin{pmatrix} 2xe^x \\ x^2e^x/2 + xe^x/2 \end{pmatrix}$$

summary!

what we did today:

- diagonalization method

plan for next time:

- Fourier series