Math 309 Lecture 13

Hilbert Space

W.R. Casper

Department of Mathematics
University of Washington

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Today!

Plan for today:

- Inner Products and Orthogonality
- Hilbert Space

Next time:

- Fourier Series

Outline

1. Inner Product Spaces
   - Inner Products
   - Orgonality and Linear Independence

2. Hilbert Space
   - Basic Definition
   - Orthogonal Basis

The Dot Product for $\mathbb{R}^n$

The dot product of two vectors $\vec{u}, \vec{v}$ is given by

$$\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = u_1v_1+u_2v_2+\cdots+u_nv_n, \text{ for } \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

For example in $\mathbb{R}^2$:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4, 2 \\ = \end{pmatrix} 2\cdot 4 + 3\cdot 2 = 14.$$

Properties of the Dot Product

The dot product has several important properties.

Symmetry

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

Linearity

$$(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) \text{ and } (\vec{u} + \vec{w}) \cdot \vec{v} = \vec{u} \cdot \vec{v} + \vec{w} \cdot \vec{v}$$

Positive Definiteness

$$\vec{u} \cdot \vec{u} \geq 0$$

with equality if and only if $\vec{u} = \vec{0}$.

Definition

A multiplication operation which takes two vectors and returns a scalar and satisfies the above three properties is called a **inner product**.

Inner Product Space

Definition

A vector space equipped with an inner product (eg. $\mathbb{R}^n$) is called a **inner product space**.

- The inner product of two vectors $\vec{u}, \vec{v}$, such as the dot product for $\mathbb{R}^n$, is denoted by $\langle \vec{u}, \vec{v} \rangle$.
- Inner products measure *angles between vectors* and *sizes of vectors*

Definition

The **magnitude** of a vector is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$. The **angle** θ between two vectors $\vec{u}, \vec{v}$ is determined by

$$\cos(\theta) = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}.$$

Pairwise othogonal

Let V be an inner product space, such as $\mathbb{R}^n$.

Definition

Two nonzero vectors $\vec{u}, \vec{v}$ are called **orthogonal** if $\vec{u} \cdot \vec{v} = 0$ (equivalently if the angle between them is 90 degrees). A collection of nonzero vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ in V is called **pairwise orthogonal** if $\vec{v}_i \cdot \vec{v}_j = 0$ for all $1 \leq i, j \leq m$ with $i \neq j$.

Example

The collection of vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is pairwise orthogonal.

Pairwise orthogonal

Example

The collection of vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is pairwise orthogonal.

Example

The collection of vectors

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

is *not* pairwise orthogonal.

Linear Independence

Proposition

A collection of pairwise orthogonal vectors must be linearly independent.

Definition

A **orthogonal basis** for V is a collection of vectors which are pairwise orthogonal and span V. An orthogonal basis is called **orthonormal** if all of the elements are also unit vectors.

Linear Independence

Example

The collection of vectors

$$\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$$

is an orthogonal basis for $\mathbb{R}^2$. The collection of vectors

$$\left\{\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\right\}$$

is an orthonormal basis for $\mathbb{R}^2$.

Hilbert Space

Definition

A **Hilbert space** is an inner product space V (ie. a vector space with an inner product) satisfying the additional property that if $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots$ is a sequence of vectors in V and the sum $\sum_{n=1}^{\infty} \|\vec{v}_n\| < \infty$, then the sum $\sum_{n=1}^{\infty} v_n$ converges to a vector in V.

- In a Hilbert space, we can add up infinitely many vectors!
- $\mathbb{R}^n$ is an example of a Hilbert space
- We will see that the collection $\mathcal{P}_T$ of all square integrable functions with period T is also a Hilbert space

Our Favorite Hilbert Space

In this class, our favorite Hilbert space will be the collection $\mathcal{P}_T$ of all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the follwing two properties

(1) $f(x+T) = f(x)$ for all x (ie. f is periodic with period T)

(2) the integral $\int_0^T f(x)^2 dx$ exists and is finite (ie. f is square integrable on the interval $[0, T]$).

Example

The functions $\cos(x), \cos(2x), \cos(3x)$ and $\sin(x), \sin(2x), \sin(3x)$ are 2π periodic and square-integrable on $[0, 2\pi]$, and therefore belong to $\mathcal{P}_T$.

Example

The function $\tan(x)$ is 2π-periodic, but doesn't belong to $\mathcal{P}_T$ because it is not square-integrable on $[0, 2\pi]$.

Inner product on $\mathcal{P}_T$

- In $\mathcal{P}_T$, we treat functions as vectors!
- The inner product of two funcions $f(x), g(x)$ in $\mathcal{P}_T$ is

$$\langle f, g \rangle = \int_0^T f(x)g(x)dx.$$

Basis for a Hilbert Space

Let V be a Hilbert space.

Definition

A **Hilbert space basis** for V is a (possibly infinite) collection of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots$ which are linearly independent and which satisfy the property that any vector $\vec{v}$ in V may be written as a (possibly infinite) linear combination

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots = \sum_j c_j\vec{v}_j.$$

- A Hilbert space basis is different from a vector space basis when V is in finite
- If V is finite-dimensional, the two notions of a basis are the same

Orthogonal Basis for $\mathcal{P}_T$

Theorem

The collection

$$\left\{\sin\left(\frac{2\pi mx}{T}\right), \cos\left(\frac{2\pi nx}{T}\right), \cos\left(\frac{4\pi x}{T}\right) : \substack{m=1,2,3,\ldots \\ n=0,1,2,\ldots}\right\}.$$

is an orthogonal Hilbert space basis for $\mathcal{P}_T$.

- In particular

$$\int_0^T \sin(2\pi mx/T)\cos(2\pi nx/T)dx = 0 \text{ for all } m, n$$

$$\int_0^T \cos(2\pi mx/T)\cos(2\pi nx/T)dx = 0 \text{ if } m \neq n \text{ and } T/2 \text{ otherwise}$$

$$\int_0^T \sin(2\pi mx/T)\sin(2\pi nx/T)dx = 0 \text{ if } m \neq n \text{ and } T/2 \text{ otherwise.}$$

summary!

what we did today:

- Hilbert space

plan for next time:

- Fourier series