Math 309 Lecture 14

Fourier Series

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Today!

Plan for today:

- Fourier Series

Next time:

- More Fourier Series Practice
- Convergence of Fourier Series

Outline

Big Idea

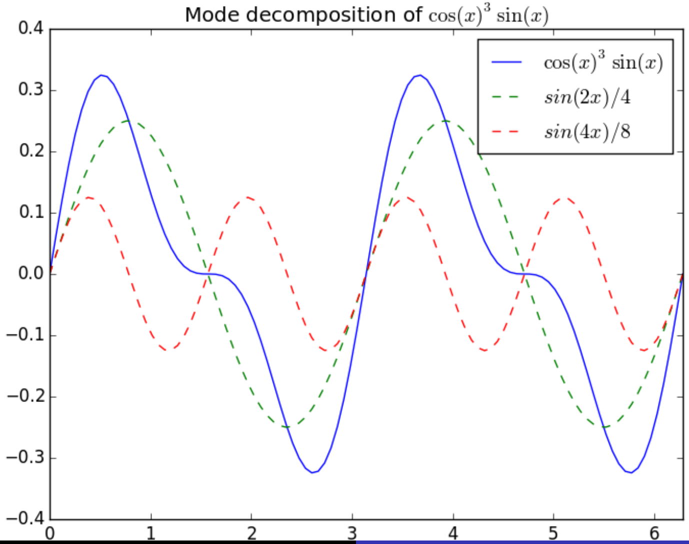
Big Idea: Any periodic motion is made up of a linear combination of “simple" periodic motions, ie. ones with pure frequencies.

- A periodic function like $f(x) = \cos(2\pi x/T)$ is easy to understand
- A periodic function like $g(x) = \cos^3(2\pi x/T)\sin(2\pi x/T)$ is more difficult!
- After a bunch of trig. identities...

$$g(x) = \frac{1}{4}\sin(4\pi x/T) + \frac{1}{8}\sin(8\pi x/T)$$

This latter expression is much more understandable. Also easier to differentiate, integrate, etc.

Graphical Interpretation



Big Idea

- What other functions can be expressed this way?
- We already know the answer!
- The simple trig functions are a Hilbert space basis for $\mathcal{P}_T$
- Everything in $\mathcal{P}_T$ can be expressed as a sum of trig functions!
- If $f(x) \in \mathcal{P}_T$ then for $L = T/2$:

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{m=1}^{\infty} b_n \sin\left(\frac{m\pi x}{L}\right).$$

Definition

For $f(x) \in \mathcal{P}_T$ the above series expansion is called the **Fourier series** of $f(x)$. The coefficients a_n and b_n are called the **Fourier coefficients** of $f(x)$.

An Example

We've already looked at a simple example with a periodic function whose Fourier series has finitely many terms. Here we'll consider the more complicated example of a “triangular wave".

Example

The Fourier series of the 2-periodic function $f(x)$ defined by

$$f(x) = |x| \text{ for } 0 \leq x \leq 1 \text{ with } f(x+2) = f(x) \text{ for all } x$$

is given by

$$\begin{aligned} f(x) &= \frac{4}{\pi^2}\cos(\pi x) + \frac{4}{9\pi^2}\cos(3\pi x) + \frac{4}{25\pi^2}\cos(5\pi x) + \dots \\ &= \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2\pi^2}\cos((2n+1)\pi x). \end{aligned}$$

An Example

Example

The Fourier coefficients of the 2-periodic function $f(x)$ defined by

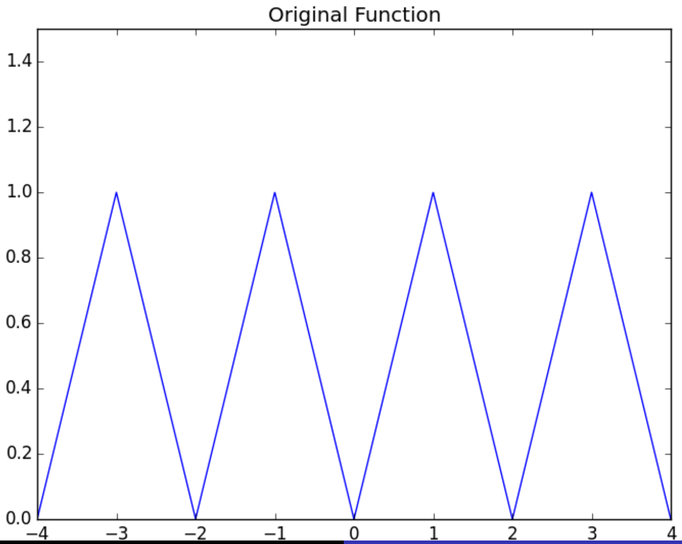
$$f(x) = |x| \text{ for } 0 \leq x \leq 1 \text{ with } f(x+2) = f(x) \text{ for all } x$$

are therefore

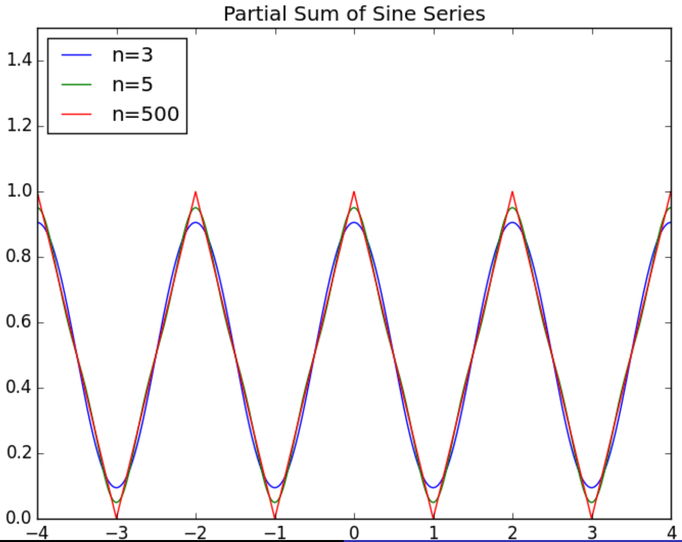
$$a_n = \begin{cases} \frac{4}{n^2\pi^2}, & n \text{ odd} \\ 1/2, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$b_m = 0 \text{ for all } m.$$

Graphical Interpretation



Graphical Interpretation



Euler-Fourier Derivation

Question

How do we calculate Fourier coefficients?

- We can use the fact that the elementary trigonometric functions are pairwise orthogonal!
- Suppose $f(x)$ has the Fourier series:

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right),$$

- Then we calulate

$$\left\langle f(x), \cos\left(\frac{j\pi x}{L}\right)\right\rangle = \sum_{n=0}^{\infty} a_n \left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{j\pi x}{L}\right)\right\rangle + \sum_{m=1}^{\infty} b_m \left\langle \sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{j\pi x}{L}\right)\right\rangle.$$

- By orthogonality, all the terms are dead, except for one $(n = j)$!

Euler-Fourier Derivation

Question

How do we calculate Fourier coefficients?

- This gives

$$\left\langle f(x), \cos\left(\frac{j\pi x}{L}\right)\right\rangle = a_j \left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{j\pi x}{L}\right)\right\rangle.$$

- We also calculate

$$\left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{j\pi x}{L}\right)\right\rangle = \int_0^{2L} \cos^2\left(\frac{n\pi x}{L}\right) dx = L.$$

- Using this, we obtain (for $n > 0$):

$$a_j = \frac{1}{L}\left\langle f(x), \cos\left(\frac{j\pi x}{L}\right)\right\rangle = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{j\pi x}{L}\right) dx.$$

- Can similarly derive an equations for a_0 and b_j.

Euler-Fourier Formulas

- Together, these give us the Euler-Fourier formulas!

Theorem

Euler-Fourier Let $f(x)$ be in $\mathcal{P}_T$. Then the Fourier coefficients of $f(x)$ are given for $n > 0$ by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{j\pi x}{L}\right)dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{j\pi x}{L}\right).$$

Furthermore, the 0'th coefficient a_0 is given by $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$ and is equal to the average value of $f(x)$.

Exercise

Try these in class:

Question

Use the Euler-Fourier formulas to derive the Fourier coefficients we obtained for the triangular wave above. (Note in this case $T = 2$ so $L = 1$).

Question

Use the Euler-Fourier formulas to derive the Fourier coefficients for the square wave:

$$f(x) = \begin{cases} 1 & 0 \leq x < 3 \\ 0 & -3 \leq x < 0 \end{cases}$$

with $f(x+6) = f(x)$ for all x.

summary!

what we did today:

- Fourier series

plan for next time:

- More Fourier series practice
- Convergence of Fourier series