Math 309 Lecture 15

Fourier Series Convergence

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Today!

Plan for today:

- Fourier Series Convergence

Next time:

- Sine and Cosine Series

Outline

1. Convergence of Fourier Series
   - Recap of Last Time
   - Convergence
   - Piecewise Continuity and Pointwise Convergence
   - Square Wave Example

Periodic Functions

Recall that $\mathcal{P}_T$ is the Hilbert space of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the two properties

(1) $f(x + T) = f(x)$ for all x (ie. f is periodic with period T)

(2) the integral $\int_0^T f(x)^2 dx$ exists and is finite (ie. f is square integrable on the interval $[0, T]$).

Given a function $f(x)$ in $\mathcal{P}_T$, we can use the Euler-Fourier formulas to express (for $L = T/2$)

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right),$$

Euler-Fourier Formulas

where here $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$ and for $n > 0$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

These are the Euler-Fourier Formulas.

Meaning of Convergence?

- What does the expression

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)$$

  actually mean?
- The appearance of ∞ means we are taking a **limit**, so the above is a statement about **convergence.**
- In other words, we're looking at the **partial Fourier sums**

$$S_N(x) = \sum_{n=0}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{m=1}^{N} b_m \sin\left(\frac{m\pi x}{L}\right),$$

- and asking about what happens as we choose N bigger and bigger ie.

$$\lim_{N\to\infty} S_N(x).$$

Convergence in What Sense?

- Whenever you see a statement about convergence, immediately ask

  convergence in what sense?
- There are many types of convergence, such as:
  (1) uniform convergence
  (2) pointwise convergence
  (3) L^2-convergence (ie. Hilbert space norm convergence)
- pointwise convergence means for every fixed value of x, $S_N(x) \to f(x)$
- uniform convergence is stronger: it implies pointwise convergence
- L^2-convergence doesn't quite imply pointwise convergence

Hilbert Space Norm Convergence

- We derived the Fourier coefficients using Hilbert space inner products, so we're only guaranteed L^2-convergence
- This means that

$$\lim_{N\to\infty}\int_{-L}^{L}|S_N(x)-f(x)|^2dx \to 0$$

- In practice, this means that $S_N(x)$ converges pointwise to $f(x)$ **most places**
- Technically everywhere but a **set of measure zero**, such as finitely many points

Piecewise Continuity

- If we also assume $f(x)$ is sufficiently nice, we can get a pointwise convergence result!

Definition

We call a periodic function f in $\mathcal{P}_T$ **piecewise continuous** if f is continuous everywhere on $[-L, L]$, except for finitely many points.

Example

The square wave defined by $f(x+2) = f(x)$ for all x, with $f(x) = 0$ for $-1 \leq x < 0$ and $f(x) = 1$ for $0 \leq x < 1$ is piecewise continuous, since it has discontinuities only at 0 and 1 in $[-1, 1]$.

Piecewise Continuity

Example

The triangular wave from last lecture is continuous, and therefore piecewise continuous

Example

The periodic function $f(x) = \tan(x)$ has period π and is piecewise continuous, since it has discontinuities only at $\pm\pi/2$ in $[-\pi/2, \pi/2]$. However, it is not in $\mathcal{P}_\pi$ because it is not square integrable.

Example

The function $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational is periodic with period 1, but is not piecewise continuous.

Pointwise Convergence

- For piecewise continuous functions, we have an amazing convergence result!

Theorem (Pointwise Convergence)

Suppose that $f(x)$ is in $\mathcal{P}_T$ and is piecewise continuous. Let a_n, b_m be the Fourier coefficients of $f(x)$ for $n \geq 0$ and $m > 0$. Then for every fixed point p,

$$\frac{f(p+) + f(p-)}{2} = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi p}{L}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi p}{L}\right)$$

where here $f(p+)$ and $f(p-)$ are the left and right limits of $f(x)$ as $x \to p$.

Square Wave Example

- Consider the square wave

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & -1 \le x < 0 \end{cases}, \quad \text{with } f(x+2) = f(x) \text{ for all } x.$$

- Using the Euler-Fourier formulas (with $L = 1$) we get:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_0^1 1dx = \frac{1}{2}.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_0^1 \cos(n\pi x)dx = \frac{1}{n\pi}\sin(n\pi x)|_0^1 = 0.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^1 \sin(n\pi x)dx$$

$$= \frac{-1}{n\pi}\cos(n\pi x)|_0^1 = \frac{1-\cos(n\pi)}{n\pi} = \frac{1-(-1)^n}{n\pi}.$$

Square Wave Example

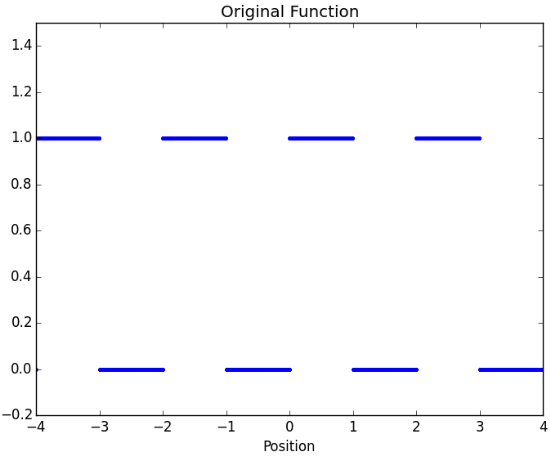
- Note that $f(x)$ is continuous for $x \neq \dots, -2, -1, 0, 1, 2, \dots$
- Therefore $f(x+) = f(x-) = f(x)$ for x not an integer
- Also $f(0+) = 1$, $f(0-) = 0$, $f(1+) = 0$, $f(1-) = 1$, etc.
- Therefore

$$\frac{f(x+) + f(x-)}{2} = \begin{cases} f(x) & \text{if } x \text{ is an integer} \\ 1/2 & \text{otherwise} \end{cases}$$

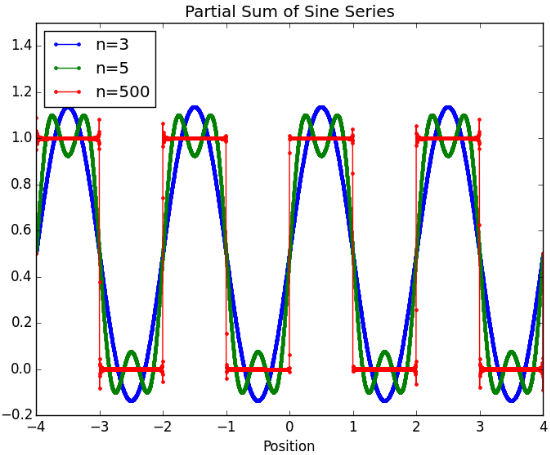
- In particular for any non-integer p

$$f(p) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{1 - (-1)^n}{n\pi} \sin\left(\frac{m\pi p}{L}\right)$$

Graphical Interpretation



Graphical Interpretation



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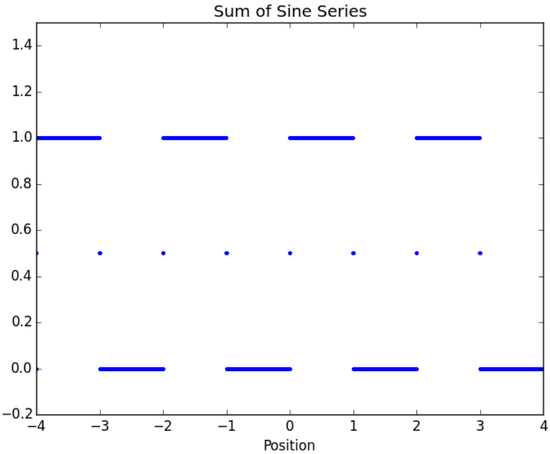

Exercises

Example

Work out the Fourier series for the function

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 3 & 1 \leq x < 2 \\ 2 & 2 \leq x < 3 \end{cases} \quad \text{with } f(x+3) = f(x) \text{ for all } x.$$

What does the Fourier series converge to for each value of x? Where does it converge to the original value of the function $f(x)$?

summary!

what we did today:

- Fourier series convergence

plan for next time:

- Sine and Cosine Series