# Math 309 Lecture 16

## Sine and Cosine Series

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# Today!

Plan for today:

- Sine and Cosine Series

Next time:

- Boundary Value Problems

# Outline

# Expanding Arbitrary Functions

- So far, we've used Fourier series to expand trig functions in terms of elementary trig functions
- What about functions $f(x)$ which are not periodic, such as $f(x) = x^2$?
- We can – but only on a fixed finite interval $[0, L]$, where we get to pick $L$.
- To do so, we replace $f(x)$ with a periodic function agreeing with $f(x)$ on $[0, L]$
- Then we take the Fourier transform

# An Example

## Example

Consider $f(x) = x^2$. We can find an expansion for $f(x)$ in terms of elementary trig functions on $[0, L]$ by replacing $f(x)$ with $g(x)$ defined by

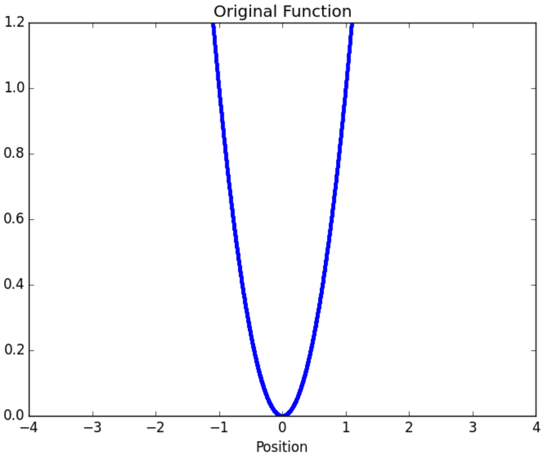
$$g(x) = f(x) - L \leq x < L, \quad \text{with } g(x + 2L) = g(x) \text{ for all } x.$$

Using the Euler-Fourier equations to get the Fourier coefficients, we find

$$g(x) = \frac{-L^3}{6} + \sum_{n=1}^{\infty} \frac{4L^3(-1)^n}{\pi^2 n^2} \cos\left(\frac{n\pi x}{L}\right)$$

since $g(x)$ agrees with $f(x)$ on $[0, L]$, this series converges to $f(x)$ there too!

# Graphical Interpretation



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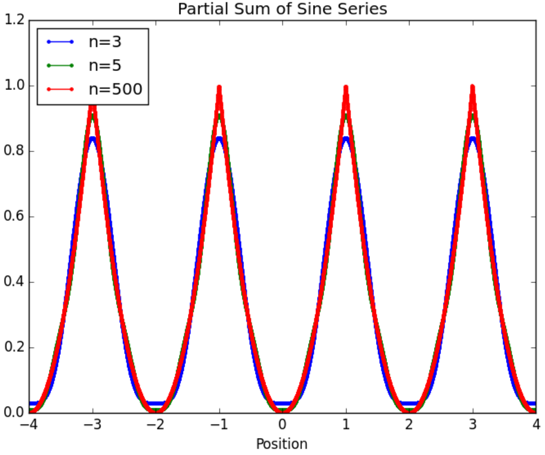

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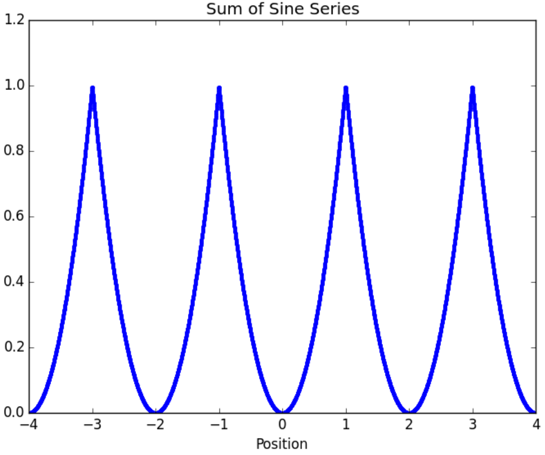


# Which Periodic Expansion?

- Notice that the periodic expansion we chose of $f(x) = x^2$ was not unique!
- Other options include

$$g(x) = \begin{cases} x^2 & 0 \leq x < L \\ -x^2 & -L \leq x < 0 \end{cases} \quad \text{with } g(x+2L) = g(x) \text{ for all } x.$$

$$g(x) = \begin{cases} x^2 & 0 \leq x < L \\ 0 & -L \leq x < 0 \end{cases} \quad \text{with } g(x+2L) = g(x) \text{ for all } x.$$

$$g(x) = \begin{cases} x^2 & 0 \leq x < L \\ -x & -L \leq x < 0 \end{cases} \quad \text{with } g(x+2L) = g(x) \text{ for all } x.$$

- These are all periodic and agree with $f(x)$ on $[0, L]$.

# Which Periodic Expansion?

- So which expansion should we choose?
- Here's some points to keep in mind
  (1) Our pointwise convergence theorem gives pointwise convergence, but **doesn't say what rate**. Some periodic extensions will give faster convergence!
  (2) In applications, there may be a natural choice of periodic extension
  (3) If we extend it to a **even** or **odd** periodic function, then our Fourier series will involve **sine** terms or **cosine** terms only.
- We'll focus on this last point...

# Sine and Cosine Series

## Definition

Let $f(x)$ be a function defined on an interval $[0, L]$. The **cosine series** for $f(x)$ is an expansion of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_0 \cos\left(\frac{n\pi x}{L}\right).$$

The **sine series** for $f(x)$ is an expansion of the form

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right).$$

# Calculating Cosine Series

- To calculate the $a_n$'s in the cosine series:
  (1) Expand $f(x)$ evenly and periodically (with period $2L$), getting a new function $g(x)$
  (2) Calculate the Fourier coefficients of $g(x)$
  (3) The $b_m$'s in the Fourier series will all be zero, and the $a_n$'s will be the coefficients we want!

# Calculating Sine Series

- To calculate the $b_n$'s in the sine series:
  1. Expand $f(x)$ oddly and periodically (with period $2L$), getting a new function $g(x)$
  2. Calculate the Fourier coefficients of $g(x)$
  3. The $a_n$'s in the Fourier series will all be zero, and the $b_m$'s will be the coefficients we want!

# An Example

**Example**

Calculate the cosine series of the function $f(x) = 2x$ on the interval $[0, 3]$.

- We want a cosine series, so we expand $f(x)$ evenly to a function $g(x)$:

$$g(x) = \begin{cases} 2x & 0 \leq x < 3 \\ -2x & -3 \leq x < 0 \end{cases} \quad \text{with } g(x+6) = g(x) \text{ for all } x.$$

# An Example

- Using this, we calculate (with even/odd arguments)

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(x)dx = \frac{1}{L}\int_0^L g(x)dx = \frac{1}{3}\int_0^3 2xdx = \frac{9}{3}.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{3}\int_0^3 2x\cos\left(\frac{n\pi x}{3}\right) = \frac{12((-1)^n - 1)}{n^2\pi^2}$$

$$b_m = \frac{1}{L}\int_{-L}^{L} g(x)\sin\left(\frac{m\pi x}{L}\right)dx = 0.$$

# An Example

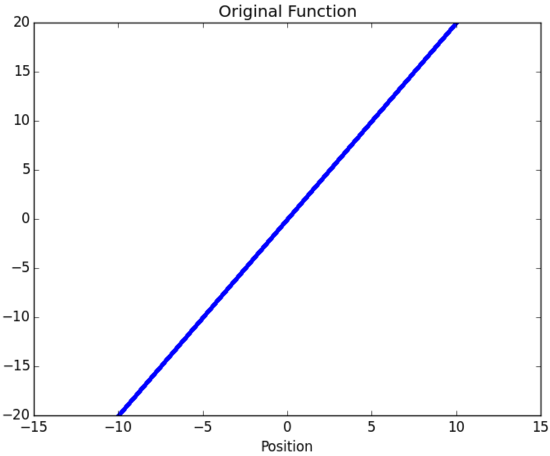
- This tells us

$$g(x) = \frac{9}{3} + \sum_{n=1}^{\infty} \frac{12((-1)^n - 1)}{n^2\pi^2} \cos\left(\frac{n\pi x}{3}\right),$$

and therefore for $0 \le x < 3$:

$$f(x) = 2x = \frac{9}{3} + \sum_{n=1}^{\infty} \frac{12((-1)^n - 1)}{n^2\pi^2} \cos\left(\frac{n\pi x}{3}\right),$$

# Graphical Interpretation



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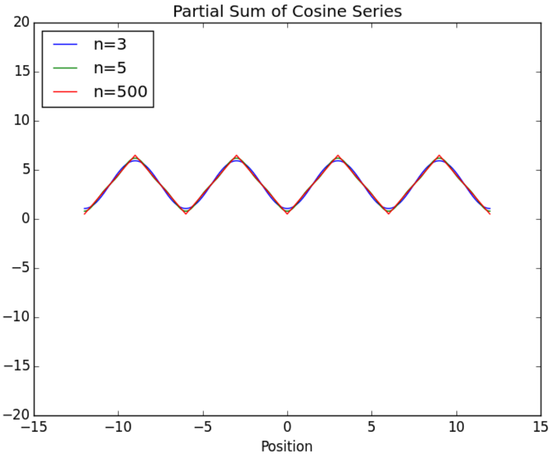

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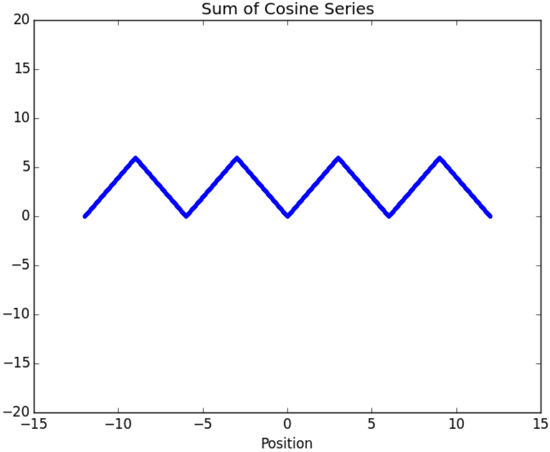


# An Example

**Example**

Calculate the sine series of the function $f(x) = 1 - x$ on the interval $[0, 1]$.

- We want a sine series, so we expand $f(x)$ oddly to a function $g(x)$:

$$g(x) = \begin{cases} 1 - x & 0 \leq x < 1 \\ -1 - x & -3 \leq x < 0 \end{cases} \quad \text{with } g(x+2) = g(x) \text{ for all } x.$$

# An Example

- Using this, we calculate (with even/odd arguments)

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(x)dx = 0.$$

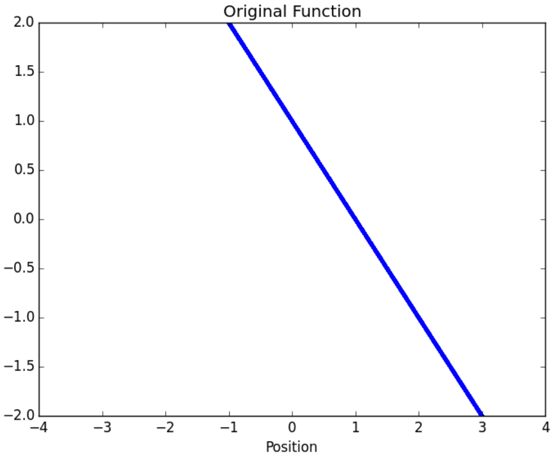
$$a_n = \frac{1}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right) dx = 0.$$

$$b_m = \frac{1}{L}\int_{-L}^{L} g(x)\sin\left(\frac{m\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} g(x)\sin\left(\frac{m\pi x}{L}\right) dx$$
$$= 2\int_0^1 (1-x)\sin(m\pi x) = \frac{2}{m\pi}.$$

# An Example

- This tells us
$$g(x) = \sum_{m=1}^{\infty} \frac{2}{m\pi} \sin(m\pi x),$$
and therefore for $0 \leq x < 1$:
$$f(x) = 1 - x = \sum_{m=1}^{\infty} \frac{2}{m\pi} \sin(m\pi x).$$

# Graphical Interpretation



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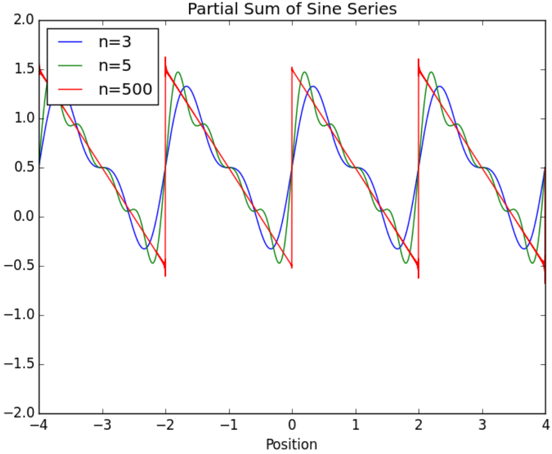

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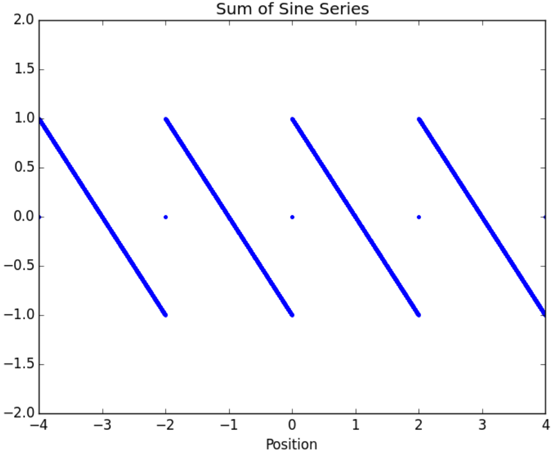

# summary!

what we did today:

- Sine and Cosine Series

plan for next time:

- Boundary Value Problems