Math 309 Lecture 4

Linear Independence and the Wronskian

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Today!

Plan for today:

- Wronskian
- More Practice with Homogeneous First Order Linear Systems

Next time:

- Direction Fields (eg. Slope Fields)

Outline

Linear Dependence and Independence

- We first define the notion of linear dependence and independence for functions

Definition

A collection of vector-valued functions $\{\vec{f}_1(x), \dots, \vec{f}_m(x)\}$ is **linearly dependent on the interval** (α, β) if there exist constants $c_1, c_2, \dots, c_m$ with at least one of the constants nonzero, such that

$$c_1\vec{f}_1(x) + c_2\vec{f}_2(x) + \cdots + c_m\vec{f}_m(x) = \vec{0}$$

for all $x \in (\alpha, \beta)$. If the collection of functions is not linearly dependent, then it is called **linearly independent.**

The Wronskian

Question

Given a collection of functions $\{\vec{f}_1(x), \dots, \vec{f}_m(x)\}$, how can we check whether their linearly dependent or independent on an interval (α, β)?

- Can't really try *all* values of x (infinitely many!)
- If the vector functions each have n entries and $m = n$, we have a tool that can help!

Definition

The **Wronskian** of n functions $\vec{f}_1(x), \dots, \vec{f}_n(x)$ from $\mathbb{R}$ to $\mathbb{R}^n$ is defined to be

$$W[\vec{f}_1(x), \vec{f}_2(x), \dots, \vec{f}_n(x)] = \det([\vec{f}_1(x)\ \vec{f}_2(x)\ \dots\ \vec{f}_n(x)]).$$

Example 1

- Consider the functions

$$\vec{f}_1(x) = \begin{pmatrix} 2x \\ x \end{pmatrix}, \quad \vec{f}_2(x) = \begin{pmatrix} e^x \\ e^{-x} \end{pmatrix}.$$

- We calculate the Wronskian:

$$W[\vec{f}_1(x), \vec{f}_2(x)] = \det\left(\begin{bmatrix} 2x & e^x \\ x & e^{-x} \end{bmatrix} \right) = 2xe^{-x} - xe^x.$$

Example 2

- Consider the functions

$$\vec{f}_1(x) = \begin{bmatrix} e^x \\ 2e^x \\ 3e^x \end{bmatrix}, \quad \vec{f}_1(x) = \begin{bmatrix} e^{2x} \\ 2e^{2x} \\ 4e^{2x} \end{bmatrix}, \quad \vec{f}_3(x) = \begin{bmatrix} e^{-x} \\ -e^{-x} \\ e^{-x} \end{bmatrix}.$$

- We calculate the Wronskian:

$$W[\vec{f}_1(x), \vec{f}_2(x), \vec{f}_3(x)] = \det\left(\begin{bmatrix} e^x & e^{2x} & e^{-x} \\ 2e^x & 2e^{2x} & -e^{-x} \\ 3e^x & 4e^{2x} & e^{-x} \end{bmatrix}\right) = 3e^{2x}.$$

Wronskian Theorem 1

- The following theorem demonstrates how the Wronskian may be used to show linear dependence/independence

Theorem (Wronskian Theomem 1)

A set of functions $\vec{f}_1(x), \dots, \vec{f}_n(x)$ from $\mathbb{R}$ to $\mathbb{R}^n$ is linearly independent in the interval (α, β) if

$$W[\vec{f}_1(x), \dots, \vec{f}_n(x)] \neq 0$$

for at least one value of x in the interval (α, β)

Revisiting Example 2

- In a previous example, we considered the functions

$$\vec{f}_1(x) = \begin{bmatrix} e^x \\ 2e^x \\ 3e^x \end{bmatrix}, \quad \vec{f}_1(x) = \begin{bmatrix} e^{2x} \\ 2e^{2x} \\ 4e^{2x} \end{bmatrix}, \quad \vec{f}_3(x) = \begin{bmatrix} e^{-x} \\ -e^{-x} \\ e^{-x} \end{bmatrix}.$$

- We calculated the Wronskian:

$$W[\vec{f}_1(x), \vec{f}_2(x), \vec{f}_3(x)] = 3e^{2x}.$$

- Since this is nonzero for at least one value of x, we see that these functions are linearly independent on the interval $(-\infty, \infty)$.

Wronskian Danger

WARNING

Be careful! If the wronskian is nonzero, then we have linear independence. However, if the Wronskian is zero it DOES NOT mean that we have linear dependence.

- Consider the vector functions

$$\vec{f}_1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{f}_2(x) = \begin{pmatrix} e^x \\ 0 \end{pmatrix}.$$

- We calculate

$$W[\vec{f}_1(x), \vec{f}_2(x)] = \det\left(\begin{bmatrix} 1 & e^x \\ 0 & 0 \end{bmatrix}\right) = 0.$$

- However, the functions are linearly independent, since

$$c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} e^x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow c_1 + c_2 e^x = 0 \Rightarrow c_1 = 0, \; c_2 = 0.$$

Wronskian Theorem 2

- The previous warning is not important if the functions you are checking are solutions of the same homogeneous equation

Theorem (Wronskian Theomem 2)

Let $A(x)$ be an $n \times n$ matrix of functions whose coefficient functions are continuous on an interval (α, β). A set of solutions $\vec{y}_1(x), \dots, \vec{y}_n(x)$ to $y'(t) = A(t)y(t)$ on the interval (α, β) is linearly independent on (α, β) if and only if

$$W[\vec{y}_1(x), \dots, \vec{y}_n(x)] := \det(\vec{y}_1 \ \vec{y}_2 \ \dots \ \vec{y}_n)$$

is nonzero on some point of the interval (α, β). In this case, the Wronskian is nonzero for all $x \in (\alpha, \beta)$.

An Example

Consider the homogeneous linear system of ODE's

$$y'(x) = A(x)y(x), \quad A(x) = \begin{pmatrix} 1 & -2e^{-3x} \\ 0 & 2 \end{pmatrix}$$

Two solutions on the interval $(-\infty, \infty)$ are

$$\vec{y}_1(x) = \begin{pmatrix} e^x \\ 0 \end{pmatrix}, \quad \vec{y}_2(x) = \begin{pmatrix} e^{-x} \\ e^{2x} \end{pmatrix}$$

Is this a fundamental set of solutions on $(-\infty, \infty)$?

An Example

We check the Wronskian!

$$W[\vec{y}_1, \vec{y}_2] = \det(\vec{y}_1 \ \vec{y}_2) = \det \begin{pmatrix} e^x & e^{-x} \\ 0 & e^{2x} \end{pmatrix} = e^{3x}$$

- it is nonzero on all of $(-\infty, \infty)$
- consequently $\vec{y}_1$ and $\vec{y}_2$ are linearly independent
- since A is a 2×2 matrix, this means $\vec{y}_1, \vec{y}_2$ form a fundamental set of solutions on the interval $(-\infty, \infty)$
- general solution is therefore

$$\vec{y} = c_1 \begin{pmatrix} e^x \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} e^{-x} \\ e^{2x} \end{pmatrix}$$

Summary!

What we did today:

- Systems of first order linear ODEs

Plan for next time:

- Systems of first order ODEs