# Math 309 Lecture 9

## Diagonalization and Jordan Normal Form

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# Today!

Plan for today:

- Diagonalization
- Jordan Normal Form
- Calculating a Fundamental Matrix

Next time:

- Nonhomogeneous Differential Equations

# Outline

# Diagonalizable Matrices

- two matrices $A$ and $B$ are **similar** if there exists an invertible matrix $P$ satisfying $P^{-1}AP = B$
- natural concept – related to change of basis
- a matrix is said to be **diagonalizable** if it is similar to a diagonal matrix
- a matrix which is not diagonalizable is called **defective**

**Question**

What matrices are diagonalizable?

# Eigenbasis

- we have the following theorem:

**Theorem**

Let $A$ be a diagonal matrix. Then the following are equivalent:

(a) $A$ is diagonalizable

(b) $\mathbb{C}^n$ as a basis consisting of eigenvectors of $A$ (an **eigenbasis**)

(c) for every eigenvalue $\lambda$ of $A$, the algebraic and geometric multiplicity of $\lambda$ are the same

- in other words, $A$ needs “enough" eigenvectors

# Special cases

- there are a couple theorems that help us to decide right away if matrices are diagonalizable

**Theorem**

If all of the eigenvalues of $A$ have algebraic multiplicity 1, then $A$ is diagonalizable

**Theorem (Spectral Theorem)**

If $A$ is **normal** (ie. $A$ and $A^\dagger$ commute), then $A$ is diagonalizable

- in particular, **Hermitian** ($A = A^\dagger$) and **unitary** ($A^\dagger = A^{-1}$)) matrices are diagonalizable

# Example

- the following matrix is diagonalizable (why?):

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 4 & -1 \end{pmatrix}$$

- the following matrix is diagonalizable (why?):

$$\begin{pmatrix} 3 & 4 & 9 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{pmatrix}$$

- the following matrix is NOT diagonalizable (why?):

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

# Use the Eigenbasis

**Question**

How do we diagonalize a matrix?

- suppose that $A$ is diagonalizable
- let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be an eigenbasis for $\mathbb{R}^n$
- let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the corresponding eigenvalues (resp)
- then $P^{-1}AP = D$ for

$$P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

- order is important!!

# Example

## Question

Find $P$ invertible and $D$ diagonal so that $P^{-1}AP = D$ for

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 4 & -1 \end{pmatrix}$$

Steps:

1. Calculate the eigenvalues (diagonal values of matrix $D$)
2. For each eigenvalue, find a basis for the eigenspace
3. Put all the bases together to get an eigenbasis for $\mathbb{R}^3$
4. Use them as column vectors in matrix $P$

# Example

1. characteristic poly:
$$p_A(x) = \det(A - xI) = -(x+3)(x-2)(x-8)$$
Therefore eigenvalues are $-3, 2, 8$
2. corresponding eigenspaces:
$$E_{-3} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$$
$$E_2 = \operatorname{span}\left\{\begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}\right\}$$
$$E_8 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\}$$

# Example

3. eigenbasis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

4. consequently we have

$$P = \begin{pmatrix} 0 & -5 & 1 \\ -1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

# Basic Definition

- a **Jordan block** of size $m$ is an $m \times m$ matrix of the form

$$J_m(\lambda) := \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

- ie. it is a matrix with some constant value $\lambda$ on the main diagonal and 1 on the first superdiagonal

# Examples

- some examples of **Jordan blocks** include

$$J_1(13) = (13)$$

$$J_2(-7) = \begin{pmatrix} 7 & 1 \\ 0 & 7 \end{pmatrix}$$

$$J_3(-\sqrt{5}) = \begin{pmatrix} \sqrt{5} & 1 & 0 \\ 0 & \sqrt{5} & 1 \\ 0 & 0 & \sqrt{5} \end{pmatrix}$$

$$J_4(\pi) = \begin{pmatrix} \pi & 1 & 0 & 0 \\ 0 & \pi & 1 & 0 \\ 0 & 0 & \pi & 1 \\ 0 & 0 & 0 & \pi \end{pmatrix}$$

# Jordan Normal Form

- A matrix $B$ is in **Jordan normal form** if it is in the form

$$B = \begin{pmatrix} J_{m_1}(\lambda_1) & 0 & \dots & 0 \\ 0 & J_{m_2}(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{m_\ell}(\lambda_\ell) \end{pmatrix}$$

- for example, a diagonal matrix is a matrix in Jordan normal form
- other examples include

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

# Jordan Decomposition

- not every matrix is diagonalizable
- however, *every* matrix is similar to a matrix in Jordan normal form
- the Jordan normal form of a matrix is unique (up to permutation of the Jordan blocks)
- one way to think about this is in terms of **generalized eigenvectors**
- a **generalized eigenvector of rank** $k$ with eigenvalue $\lambda$ is a nonzero vector in the kernel of $(A - \lambda I)^k$ but not in the kernel of $(A - \lambda I)^{k-1}$
- the dimension of the space of generalized eigenvectors of an eigenvalue is always the same as the algebraic multiplicity
- this gives rise to Jordan normal form

# How to Decompose

## Question

How do we find $P, N$ so that $P^{-1}AP = N$, with $N$ in Jordan normal form?

- difficult operation in general
- for each eigenvalue $\lambda$, find a basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ of the eigenspace $E_\lambda(A)$
- then find generalized eigenvectors...
- difficult/long to do in general
- will focus on $2 \times 2$ and $3 \times 3$ cases

# Possible $2 \times 2$ Jordan Normal Forms

**Question**

If $A$ is a $2 \times 2$ matrix, what are the possible Jordan normal forms of $A$?

- if $A$ is nondegenerate, then $A$ is diagonalizable with eigenvalues $\lambda_1, \lambda_2$ and its Jordan form is

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

- if $A$ is degenerate, then $A$ has exactly one eigenvalue $\lambda$ with alg. mult 2, and geom. mult 1, and its Jordan form is

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

# Calculating the matrix $P$

**Question**

If $A$ is a $2 \times 2$ matrix, how do we find $P$ so that $P^{-1}AP = N$?

- if $A$ is nondegenerate, with eigenvalues $\lambda_1, \lambda_2$ we do the usual thing:

STEP 1: choose $\vec{v}_1 \in E_{\lambda_1}(A)$
STEP 2: choose $\vec{v}_2 \in E_{\lambda_2}(A)$
STEP 3: set $P = [\vec{v}_1 \ \vec{v}_2]$

- if $A$ is degenerate with eigenvalue $\lambda$, then use the following steps:

STEP 1: choose $\vec{v} \notin E_{\lambda}(A)$
STEP 2: set $\vec{u} = (A - \lambda I)\vec{v}$
STEP 3: set $P = [\vec{u} \ \vec{v}]$ (order is important!!)

# Example 1

## Question

Find the Jordan normal form of the matrix

$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

- char. poly is $(x-1)^2$, so eigenvalues are $1, 1$
- eigenspace:

$$E_1(A) = \text{span}\{\vec{v}\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

- degenerate! since alg mult $\neq$ geom mult.
- Choose $\vec{v} = \binom{0}{1} \notin E_1(A)$. Calculate $\vec{u} = (A - 1I)\vec{v} = \binom{1/2}{0}$
- then take $P = [\vec{u}\ \vec{v}]$.

# Example 1 Continued

## Question

Find the Jordan normal form of the matrix

$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

- in other words

$$P = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

- then Jordan form for $A$ is

$$N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- and we have $P^{-1}AP = N$

# Example 2

## Question

Find the Jordan normal form of the matrix

$$A = \begin{pmatrix} 7 & 1 \\ -1 & 5 \end{pmatrix}$$

- char. poly is $(x-6)^2$, so eigenvalues are $6, 6$
- eigenspace:

$$E_6(A) = \text{span}\{\vec{v}\} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

- degenerate! since alg mult $\neq$ geom mult.
- Choose $\vec{v} = \binom{1}{0} \notin E_6(A)$. Calculate $\vec{u} = (A - 6I)\vec{v} = \binom{7}{-1}$.
- then take $P = [\vec{u}\ \vec{v}]$.

# Example 2

## Question

Find the Jordan normal form of the matrix

$$A = \begin{pmatrix} 7 & 1 \\ -1 & 5 \end{pmatrix}$$

- in other words

$$P = \begin{pmatrix} 7 & -1 \\ 1 & 0 \end{pmatrix}$$

- then Jordan form for $A$ is

$$N = \begin{pmatrix} 6 & 1 \\ 0 & 6 \end{pmatrix}$$

- and we have $P^{-1}AP = N$

# summary!

what we did today:

- diagonalizable matrices
- jordan normal form

plan for next time:

- $3 \times 3$ Jordan normal form
- calculating a fundamental matrix
- nonhomogeneous equations