Math 309 Quiz 1 Practice Solutions

April 6, 2017

Problem 1. For each of the following, write TRUE if the statement is true and BANANAS if the statement is false. Throughout A is an $n \times n$ matrix.

(a) The geometric multiplicity of an eigenvalue λ of A is always larger than the algebraic multiplicity.

(b) The sum of the algebraic multiplicities of all of the eigenvalues of A is n.

(c) The geometric multiplicity of an eigenvalue λ of A is equal to the nullity of $A - \lambda I$.

(d) A square matrix is singular if and only if its determinant is nonzero.

Solution 1. (a) BANANAS

(b) TRUE

(c) TRUE

(d) BANANAS

Problem 2. For the given matrix A, determine the following

- The eigenvalues of A
- The geometric and algebraic multiplicity of each eigenvalue of A
- A basis for the eigenspace of each eigenvalue

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Solution 2. The characteristic polynomial is $x^2 - 5x - 2$, and therefore the eigenvalues are

$$\lambda_{\pm} = \frac{5 \pm \sqrt{33}}{2}$$

In particular there are two eigenvalues, both with algebraic multiplicity one and hence with geometric multiplicity also 1.

Our eigenvector trick tells us that an eigenvector with eigenvalue $\lambda_\pm$ is given by

$$\vec{v}_\pm = \begin{pmatrix} \frac{5\pm\sqrt{33}}{2} - 4 \\ 3 \end{pmatrix}.$$

In summary, we have the following table:

λ	$m_a(\lambda)$	$m_g(\lambda)$	$E_\lambda(A)$ basis
$\frac{5+\sqrt{33}}{2}$	1	1	$\left\{ \begin{pmatrix} \frac{5+\sqrt{33}}{2} - 4 \\ 3 \end{pmatrix} \right\}$
$\frac{5-\sqrt{33}}{2}$	1	1	$\left\{ \begin{pmatrix} \frac{5-\sqrt{33}}{2} - 4 \\ 3 \end{pmatrix} \right\}$

Problem 3. Give an example of a 2×2 matrix A which is degenerate (ie. has an eigenvalue whose algebraic and geometric multiplicities do not agree).

Solution 3. An example is

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$