Math 309 Quiz 2 Practice Solutions

April 13, 2017

Problem 1. TRUE or BANANAS? A diagonal matrix is also a matrix in Jordan normal form.

Solution 1. True! It's made up of 1×1 Jordan blocks!

Problem 2. Give an example of a 2×2 Jordan block.

Solution 2.

$$\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

Problem 3. Give an example of a 3×3 Jordan block.

Solution 3.

$$\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

Problem 4. Give two different examples of 4×4 matrices in Jordan normal form.

$$\begin{pmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

Problem 5. Find a basis for the solution space of the system of differential equations.

$$y_1' = 4y_1 + 2y_2$$
$$y_2' = 2y_1 + 4y_2$$

Be sure to carefully explain why what you found is a *basis*.

Solution 4. We rewrite this in matrix form as

$$\frac{d}{dt}\vec{y} = A\vec{y}, \quad A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

The eigenvalues of A are 6 and 2, and the associated eigenspaces are

$$E_6(A) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \quad E_2 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}.$$

Therefore we get two solutions

$$\vec{y}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t} = \begin{pmatrix} e^{6t} \\ e^{6t} \end{pmatrix}$$

$$\vec{y}_2(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}.$$

The functions $\vec{y}_1(t)$ and $\vec{y}_2(t)$ are linearly independent, because the Wronskian

$$W[\vec{y}_1, \vec{y}_2] = \det \begin{pmatrix} e^{6t} & e^{2t} \\ e^{6t} & -e^{2t} \end{pmatrix} = -2e^{8t}$$

is nonzero. Since the dimension of the solution space is two-dimensional, we conclude that $\{\vec{y}_1, \vec{y}_2\}$ is indeed a basis for the solution space.