Math 309 Quiz 3 Practice

April 27, 2017

Problem 1. TRUE or BANANAS? The matrix exponential $\exp(A(t)t)$ is a fundamental matrix for the system of equations

$$\frac{d}{dt}\vec{y}(t) = A(t)\vec{y}(t).$$

Solution 1. BANANAS! For this to work $A(t)$ needs to be a constant.

Problem 2. TRUE or BANANAS? If $\Psi(t)$ and $\Phi(t)$ are two fundamental matrices for the system of equations

$$\frac{d}{dt}\vec{y}(t) = A(t)\vec{y}(t),$$

then there exists an invertible constant matrix C such that $\Psi(t) = \Phi(t)C$.

Solution 2. TRUE!

Problem 3. Suppose that A is a 2×2 matrix with complex eigenvalues $a \pm ib$ (with $b \neq 0$). Determine for which values of a and b the critical point at the origin of the system

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t)$$

is spirally stable or spirally unstable.

Solution 3. If $a > 0$ it's spirally unstable, otherwise it's spirally stable.

Problem 4. Find a real basis for the solution space of the system of differential equations

$$\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= -y_1 + 2y_2 \end{aligned}$$

Solution 4.
SOLUTION METHOD I:
The eigenvalues are $2 \pm i$, with associated eigenspaces

$$\mathcal{E}_{2+i}(A) = \text{span}\left\{ \begin{pmatrix} i \\ -1 \end{pmatrix} \right\}.$$

$$\mathcal{E}_{2-i}(A) = \text{span}\left\{ \begin{pmatrix} -i \\ -1 \end{pmatrix} \right\}.$$

Using this, we have the solutions

$$\vec{y}(t) = c_1 \begin{pmatrix} i \\ -1 \end{pmatrix} e^{(2+i)t} + c_2 \begin{pmatrix} -i \\ -1 \end{pmatrix} e^{(2-i)t}.$$

However, these are not *real* solutions. To get real solutions, we can take appropriate linear combinations. Doing so, we obtain two linearly independent real solutions

$$e^{2t} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, \quad e^{2t} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}.$$

These form a real basis. **SOLUTION METHOD II:**
The eigenvalues are $2 \pm i$. Then via the matrix exponential tricks, we have

$$\exp(At) = e^{2t}\cos(t)I + e^{2t}\sin(t)(A - 2I) = \begin{pmatrix} e^{2t}\cos(t) & e^{2t}\sin(t) \\ -e^{2t}\sin(t) & e^{2t}\cos(t) \end{pmatrix}.$$

This is a fundamental matrix, so its columns form a basis for the space of solutions. Hence

$$\begin{pmatrix} e^{2t}\cos(t) \\ -e^{2t}\sin(t) \end{pmatrix}, \quad \begin{pmatrix} e^{2t}\sin(t) \\ e^{2t}\cos(t) \end{pmatrix}$$

form a basis for the space of solutions.