Math 309 Quiz 3 Solutions

May 25, 2017

Problem 1. TRUE or BANANAS? A fundamental matrix for the system of equations

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t)$$

will necessarily be invertible.

Solution 1. TRUE.

Problem 2. TRUE or BANANAS? If the Wronskian

$$W[\vec{f}_1, \ldots, \vec{f}_N]$$

of a collection of vector-valued functions $\vec{f}_1, \ldots, \vec{f}_N$ is zero, then the functions $\vec{f}_1, \ldots, \vec{f}_N$ are linearly dependent.

Solution 2. FALSE.

Problem 3. Suppose that A is a 2×2 matrix with two distinct, *real* eigenvalues λ_1 and λ_2. Under what conditions on λ_1 and λ_2 is the critical point at the origin exponentially stable, exponentially unstable, or a saddle?

Solution 3. If λ_1 and λ_2 are both positive, then the critical point is exponentially unstable. If λ_1 and λ_2 are both negative, then the critical point is exponentially stable. If λ_1 and λ_2 have opposite signs, then the critical point is a saddle.

Problem 4. Find a real basis for the solution space of the system of differential equations

$$\begin{aligned} y_1' &= 3y_1 + y_2 \\ y_2' &= -y_1 + 3y_2 \end{aligned}$$

Solution 4. The eigenvalues of the associated matrix A are $3 \pm i$. Therefore a fundamental matrix for this system is given by the matrix exponential

$$\begin{aligned} \exp(At) &= Ie^{3t}\cos(t) + (A - 3I)e^{3t}\sin(t) \\ &= \begin{pmatrix} e^{3t}\cos(t) & e^{3t}\sin(t) \\ -e^{3t}\sin(t) & e^{3t}\cos(t) \end{pmatrix} \end{aligned}$$

The colums of this form a fundamental set of solutions, ie. a basis for the space of solutions. Thus a basis is

$$\left\{ \begin{pmatrix} e^{3t}\cos(t) \\ -e^{3t}\sin(t) \end{pmatrix}, \begin{pmatrix} e^{3t}\sin(t) \\ e^{3t}\cos(t) \end{pmatrix} \right\}.$$