Math 309 Quiz 4 Solution

May 25, 2017

Problem 1. TRUE or BANANAS. If $f(x) \in \mathcal{P}_T$ is even and periodic, then the coefficients of the sines (b_m's) in its Fourier series will all be 0.

Solution 1. TRUE.

Problem 2. TRUE or BANANAS. If $f(x)$ is periodic with period T and $g(x)$ is periodic with period $\widetilde{T}$, then $f(x) + g(x)$ will also be periodic.
BONUS: If TRUE, write the period of $f(x) + g(x)$. If BANANAS, give a counter-example.

Solution 2. FALSE. For example, consider $\sin(x)$ and $\sin(\pi x)$. Both are periodic, but their sum $\sin(x) + \sin(\pi x)$ is not. To see this, suppose that it was periodic with period T. Then for any integer m

$$\sin(x) + \sin(\pi x) = \sin(x + mT) + \sin(\pi x + m\pi T).$$

At $x = 0$, this would say:

$$0 = \sin(mT) + \sin(m\pi T).$$

This implies that

$$\sin(mT) = -\sin(m\pi T).$$

and by taking inverse sine:

$$\text{either} \quad mT = -m\pi T + 2\pi n \quad \text{or} \quad mT = \pi + m\pi T + 2\pi n$$

for some integer m. Either case implies that

$$T = \frac{2\pi n}{m(1 \mp \pi)}.$$

However, T is a constant! It cannot depend on m. This is a contradiction, and shows that the desired sum is not periodic.

Here's a related question: Is the product of any two periodic functions necessarily periodic? What do you think?

Problem 3. Calculate the Fourier series of

$$f(x) = \begin{cases} 0 & -L \le x < 0 \\ 1 & 0 \le x < L \end{cases} \qquad f(x+2L) = f(x).$$

Solution 3. This is simply a straight-forward Fourier series calculation, using the Euler-Fourier formulas. We have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)$$

for

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2},$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{0}^{L}\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{n\pi}\sin\left(\frac{m\pi x}{L}\right)|_0^L = 0.$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{L}\int_{0}^{L}\sin\left(\frac{m\pi x}{L}\right)dx = -\frac{1}{m\pi}\cos\left(\frac{m\pi x}{L}\right)|_0^L = \frac{(-1)^{m+1}+1}{m\pi}.$$

Thus the desired Fourier series is

$$f(x) = \frac{1}{2} + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}+1}{m\pi}\sin\left(\frac{m\pi x}{L}\right).$$