Math 309 Quiz 5 Practice

May 25, 2017

Throughout these questions $L = T/2$ and $\mathcal{P}_T$ will denote the set

$$\mathcal{P}_T = \left\{ f(x) | f(x+T) = f(x) \text{ for all } x \text{ and } \int_{-L}^{L} f(x)^2 dx < \infty \right\}.$$

Problem 1. Determine the Sine series of the function $f(x) = \sin(x) + \sin(3x)$ on the interval $[0, \pi]$.

Solution 1. This is easy! The sine series for $f(x)$ should be

$$f(x) = \sum_{m=1}^{\infty} b_m \sin(mx).$$

Since $f(x) = \sin(x) + \sin(3x)$, we should just take $b_1 = 1$ and $b_3 = 1$, and $b_m = 0$ otherwise! In other words, $f(x)$ is already written as a sine series!

Problem 2. TRUE or BANANAS: The homogeneous Dirichlet boundary value problem

$$y'' + \lambda y = 0$$

$$y(0) = 0 \quad y(L) = 0$$

will have a nontrivial solution if and only if $\lambda = \frac{n^2\pi^2}{L^2}$.

Solution 2. TRUE.

Problem 3. TRUE or BANANAS: The homogeneous mixed boundary value problem

$$y'' + \lambda y = 0$$

$$y(0) = 0 \quad y'(L) = 0$$

will have a nontrivial solution if and only if $\lambda = \frac{(n+1/2)^2\pi^2}{L^2}$.

Solution 3. TRUE.

Problem 4. Calculate the Cosine series of $f(x) = \sin(x)$ on the interval $[0, \pi]$.

Solution 4. This is an interesting calculation! We have to extend $f(x)$ evenly and periodically. Therefore $f(x) = -\sin(x)$ for $-\pi \le x < \pi$. Using the Euler-Fourier formulas, this gives

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{\pi}\int_0^\pi \sin(x)dx = \frac{2}{\pi}.$$

More generally, we calculate

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(nx)dx = \frac{2}{L}\int_0^L f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^\pi \sin(x)\cos(nx)dx.$$

To do this last integral, just use the trig identity:

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta).$$

This says

$$\sin(x)\cos(nx) = \frac{1}{2}\sin((n+1)x) + \frac{1}{2}\sin((n-1)x),$$

so the desired integral is

$$\begin{aligned} a_n &= \frac{-1}{\pi}\left(\frac{1}{n+1}\cos((n+1)x) + \frac{1}{n-1}\cos((n-1)x)\right)|_0^\pi \\ &= \frac{-1}{\pi}\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right) \\ &= \frac{(-1)^n - 1}{\pi}\frac{2n}{n^2-1} \end{aligned}$$

Note that this doesn't apply when $n = 1$ because then the denominator is zero. Therefore we must calculate that case separately. We find:

$$a_1 = \frac{1}{L}\int_{-L}^{L} f(x)\cos(x)dx = \frac{2}{L}\int_0^L f(x)\cos(x)dx = \frac{2}{\pi}\int_0^\pi \sin(x)\cos(x)dx = \frac{1}{\pi}\sin^2(x)|_0^\pi = 0.$$

Thus we conclude that

$$\sin(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{(-1)^n - 1}{\pi}\frac{2n}{n^2-1}\cos(nx).$$

What a fun formula! Remember, this only applies on the interval $[0, \pi]$!