Math 324 Quiz 1 Practice

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Problem 1. Find the mass and the center of mass of the cube $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$, with density given by $\rho = b(x^2 + y^2 + z^2)$.

Solution 1. We first calculate the mass by

$$\begin{aligned} m &= \int_0^a \int_0^a \int_0^a \rho(x,y,z) dz dy dx \\ &= b \int_0^a \int_0^a \int_0^a x^2 + y^2 + z^2 dz dy dx \\ &= b \left(\int_0^a \int_0^a \int_0^a x^2 dz dy dx + \int_0^a \int_0^a \int_0^a y^2 dz dy dx + \int_0^a \int_0^a \int_0^a z^2 dz dy dx \right) \\ &= b(a^5/3 + a^5/3 + a^5/3) = ba^5. \end{aligned}$$

We then calculate the center of mass by calculating the first moments. Note that by symmetry of the shape and the mass distribution, $M_x = M_y$. Furthermore

$$\begin{aligned} M_x &= \int_0^a \int_0^a \int_0^a y\rho(x,y,z) dz dy dx \\ &= b \int_0^a \int_0^a \int_0^a x^2 y + y^3 + yz^2 dz dy dx \\ &= b \left(\int_0^a \int_0^a \int_0^a x^2 y dz dy dx + \int_0^a \int_0^a \int_0^a y^3 dz dy dx + \int_0^a \int_0^a \int_0^a yz^2 dz dy dx \right) \\ &= b(a^6/6 + a^6/4 + a^6/6) = 7ba^6/12. \end{aligned}$$

Therefore the center of mass is $(\overline{x}, \overline{y})$ for

$$\overline{x} = M_y/m = 7a/12$$

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Problem 2. Find the volume of the tetrahedron enclosed by the coordinate planes and the tetrahedron $4x + y + z = 4$.

Solution 2. We're trying to find the volume of the region under the curve $z = 4 - 4x - y$ in the first octant. To do so, we first need to establish the shadow of our region in the xy-plane. This is a triangle, with vertices

$(1,0)$, $(0,4)$, and $(0,0)$. The hypotenuse of this triangle is parametrized by $y = -4x + 4$. The volume V is therefore

$$\begin{aligned} V &= \int_0^1 \int_0^{-4x+4} (4 - 4x - y)dydx \\ &= \int_0^1 4(-4x+4) - 4x(-4x+4) - (-4x+4)^2/2dx \\ &= \int_0^1 (-16x + 16 + 16x^2 - 16x - 8x^2 + 16x - 8)dx \\ &= 8\int_0^1 (x-1)^2 dx = 8/3 \end{aligned}$$

Problem 3. Use cylindrical coordinates to evaluate the integral

$$\int\int\int_R e^z dV$$

where R is the region enclosed by the paraboloid $z = 6 + x^2 + y^2$, the cylinder $x^2 + y^2 = 3$, and the xy-plane.

Solution 3. We need to describe our region in cylindrical coordinates. The upper surface of the region is the paraboloid $z = 6 + x^2 + y^2$, while the lower surface is the xy-plane $z = 0$. The domain of integration in the xy-plane is the disk in the xy-plane of radious $\sqrt{3}$. In cylindrical coordinates, this is described by $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq \sqrt{3}$. Furthermore, in cylindrical coordinates, the upper surface is $z = 6 + r^2$. Therefore the integral we must evaluate (not forgetting the Jacobian) is

$$\begin{aligned} \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{6+r^2} e^z r dr d\theta &= \int_0^{2\pi} \int_0^{\sqrt{3}} (e^{6+r^2} - 1) r dr d\theta \\ &= \int_0^{2\pi} \frac{1}{2}(e^{6+r^2} - r^2)|_0^{\sqrt{3}} d\theta \\ &= \pi(e^9 - e^6 - 3) \end{aligned}$$