Math 324 Quiz 1 Solution

January 24, 2017

Make sure to **show your work**! If you need additional space, please write on the back. Don't forget to **have fun**!

Problem 1. Find the volume of the tetrahedron enclosed by the coordinate planes and the tetrahedron $2x + 3y + 4z = 5$. You *must* use integrals, just quoting a general formula is worth nothing. [*Bonus: instead work out the general case, when the plane is $ax + by + cz = d$ with a, b, c, d non-negative constants and with $d > 0$*].

Solution 1. The intersection of the plane $2x + 3y + 4z = 5$ with the x, y-plane $z = 0$ is $2x + 3y = 5$. Therefore the shadow the tetrahedron casts on the x, y-plane is the region between the x-axis, the y-axis, and this line. Therefore x ranges between $0 < x < 5/2$ and for each fixed x, y ranges $0 \le y \le -2x/3 + 5/3$. To find the volume of the region R, we integrate 1:

$$V = \int\int\int_R 1dV.$$

The region is bounded below by the surface $z = 0$, and above by the surface $z = -x/2 - 3y/4 + 5/4$. Therefore

$$\begin{aligned}
V &= \int_0^{5/2} \int_0^{-2x/3+5/3} \int_0^{-x/2-3y/4+5/4} 1dzdydx \\
&= \int_0^{5/2} \int_0^{-2x/3+5/3} -x/2 - 3y/4 + 5/4dydx \\
&= \int_0^{5/2} -x(-2x/3 + 5/3)/2 - 3(-2x/3 + 5/3)^2/8 + 5(-2x/3 + 5/3)/4dx \\
&= \int_0^{5/2} \frac{1}{24}(2x - 5)^2 dx = \frac{125}{144}
\end{aligned}$$

In fact, in general if the plane forming the tetrahedron is $ax + by + cz = d$

for some a, b, c, d positive, then the volume is

$$\begin{aligned} V &= \int\int\int_R 1 dV \\ &= \int_0^{d/a}\int_0^{-ax/b+d/b}\int_0^{-ax/c-by/c+d/c} 1 dz dy dx \\ &= \int_0^{d/a}\int_0^{-ax/b+d/b} -ax/c - by/c + d/c dy dx \\ &= \int_0^{d/a}\int_0^{-ax/b+d/b} -y(ax-d+by/2)/c dy dx \\ &= \int_0^{d/a} \frac{(ax-d)^2}{2bc} dx = \frac{d^3}{6abc} \end{aligned}$$

Problem 2. Use cylindrical coordinates to evaluate the integral

$$\int\int\int_R e^{2z} dV$$

where R is the region enclosed by the paraboloid $z = 12+x^2+y^2$, the cylinder $x^2+y^2 = 4$, and the xy-plane.

Solution 2. The region of integration is $x^2+y^2 \leq 4$, and $0 \leq z \leq 12+x^2+y^2$. In cylindrical coordinates, this is $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$ and $0 \leq z \leq 12+r^2$. Therefore the integral is

$$\begin{aligned} \int_0^{2\pi}\int_0^2\int_0^{12+r^2} e^{2z} r dz dr d\theta &= \int_0^{2\pi}\int_0^2 \frac{1}{2}(e^{24+2r^2}-1) r dr d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{8}e^{24+2r^2} - \frac{r^2}{4}\right)|_{r=0}^2 d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{8}(e^{32}-e^{24})-1\right) d\theta \\ &= 2\pi\left(\frac{1}{8}(e^{32}-e^{24})-1\right) \end{aligned}$$