Math 324 Quiz 2 Practice Solutions

January 26, 2017

Problem 1. Convert the point $(4, 4, 4)$ from cartesian coordinates to spherical coordinates.

Solution 1. We calculate

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3 \cdot 16} = 4\sqrt{3},$$

as well as

$$\theta = \tan^{-1}(y/x) = \tan^{-1}(1) = \pi/4,$$

and also

$$\phi = \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2}) = \cos^{-1}(4/(4\sqrt{3})) = \cos^{-1}(1/\sqrt{3})$$

Therefore the corresponding point in spherical is $(4\sqrt{3}, \pi/4, \cos^{-1}(1/\sqrt{3}))$.

Problem 2. Convert the point $(1, \pi/4, \pi/3)$ from spherical coordinates to cartesian coordinates.

Solution 2. We calculate

$$x = \rho \sin(\phi)\cos(\theta) = 1\sin(\pi/3)\cos(\pi/4) = \frac{\sqrt{3}}{2\sqrt{2}}.$$

$$y = \rho \sin(\phi)\sin(\theta) = 1\sin(\pi/3)\sin(\pi/4) = \frac{\sqrt{3}}{2\sqrt{2}}.$$

$$z = \rho\cos(\phi) = 1\cos(\pi/3) = \frac{1}{2}.$$

Problem 3. Write down the Jacobian for spherical coordinates

Solution 3. The Jacobian is $\rho^2 \sin(\phi)$.

Problem 4. Using spherical coordinates, set up an integral to find the volume outside of the cylinder $x^2+y^2 = 1$ and inside the sphere $x^2+y^2+z^2 = 4$. **Do not evaluate it**

Solution 4. In spherical coordinates, the cylinder has the equation $\rho^2 \sin^2(\phi) = 1$, or equivalently $\rho = \csc(\phi)$. The sphere has the equation $\rho = 2$. Therefore $\csc(\phi) \leq \rho \leq 1$. Next, the sphere intersects the cylinder in two circles defined by $x^2 + y^2 = 1$ and $z = \pm\sqrt{3}$. On these circles $\rho = 2$, meaning that $\phi = \cos^{-1}(\pm\sqrt{3}/2)$, so that $\phi = \pi/6$ or $\phi = 4\pi/6$. Therefore $\pi/6 \leq \phi \leq 4\pi/6$ and the desired integral is

$$\int\int\int_R 1dV = \int_0^{2\pi} \int_{\pi/6}^{4\pi/6} \int_{\csc(\phi)}^{2} \rho^2 \sin(\phi) d\rho d\phi d\theta.$$

Problem 5. Consider the change of variables

$$u = x^2 - y^2$$

$$v = 2xy.$$

Let R be the region R in the x, y-plane defined by all (x, y) with $x^2 + y^2 \leq 1$ and $|y| \leq x$.

(a) Sketch a graph of the region R in the x, y-plane.

(b) Show that the change of variables sends R to the half-disk S in the u, v-plane defined by $u^2 + v^2 \leq 1$ and $u > 0$.

Solution 5.

(a) Your sketch should look like a pizza slice.

(b) The boundary of R is made up of three parts: two rays $y = \pm x$ with $0 \leq x \leq 1/\sqrt{2}$ and the arc of the circle of radius one which connects the two. If $|y| = x$, then $y^2 = x^2$ and therefore $u = 0$, while $v = 2x^2$. This means that the rays $y = \pm x$ with $0 \leq x \leq 1/\sqrt{2}$ is sent to the v-axis between $v = -1$ and $v = 1$.

We also calculate $u^2 + v^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = (x^2 + y^2)^2$. Therefore if $x^2 + y^2 = 1$, $u^2 + v^2 = 1$ also. This means that the circular arc between the two rays on the unit circle is sent the circular arc on the unit circle on the right hand side of the v-axis.