Math 324 Quiz 3

February 23, 2017

Problem 1. Calculate the gradient of

$$f(x, y) = ye^{-xy}.$$

Solution 1. We calculate

$$\nabla f = \langle -y^2 e^{-xy}, e^{-xy} - xye^{-xy} \rangle.$$

Problem 2. The gradient of the function $f(x, y)$ from the previous problem is $\langle -4, 1 \rangle$ at the point $(0, 2)$. Determine in which directions the directional derivative $f(x, y)$ at this point has the value 1. Remember: directions are unit vectors!

Solution 2. Suppose that $\vec{u} = \langle u_1, u_2 \rangle$ is a unit vector with $D_{\vec{u}} f = 1$ at $(0, 2)$. Then

$$1 = D_{\vec{u}} f = \langle -4, 1 \rangle \cdot \langle u_1, u_2 \rangle = -4u_1 + u_2.$$

Thus $-4u_1 + u_2 = 1$, and therefore $u_2 = 1 + 4u_1$. Since $u_1^2 + u_2^2 = 1$, this implies

$$u_1^2 + (1 + 4u_1)^2 = 1$$

This simplifies to

$$8u_1 + 17u_1^2 = 0.$$

Therefore $u_1 = 0$ or $u_1 = -8/17$. Then since $u_2 = 1 + 4u_1$, this tells us that

$$\vec{u} = \langle 0, 1 \rangle, \text{ or } \vec{u} = \langle -8/17, -15/17 \rangle$$

Problem 3. Let $f(x, y)$ and $g(x, y)$ be differentiable functions of x, y, with $g(x, y) \neq 0$. Show that the gradient satisfies a "quotient rule":

$$\nabla \left(\frac{f}{g} \right) = \frac{1}{g^2} (g \nabla f - f \nabla g).$$

Solution 3. We calculate

$$\begin{aligned} \nabla \left(\frac{f}{g} \right) &= \langle (f/g)_x, (f/g)_y \rangle \\ &= \langle (f_x g - f g_x)/g^2, (f_y g - f g_y)/g^2 \rangle \\ &= \langle f_x g - f g_x, f_y g - f g_y \rangle / g^2 \\ &= (\langle f_x, f_y \rangle g - f \langle g_x, g_y \rangle)/g^2 \\ &= ((\nabla f) g - f(\nabla g))/g^2. \end{aligned}$$